Infinity

BOOKS BY LILLIAN R. LIEBER
WITH DRAWINGS BY HUGH G. LIEBER

- Non-Euclidean Geometry
- Galois and the Theory of Groups
- The Education of T. C. Mits
- The Einstein Theory of Relativity
- Mathematics for the Two Billion
- — Mits, Wits and Logic
- Infinity

Good-bye Mr. Man, Hello Mr. NEWman
(a book of drawings)
BY HUGH G. LIEBER
with an Introduction by LILLIAN R. LIEBER

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Infinity

by

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Mathematics
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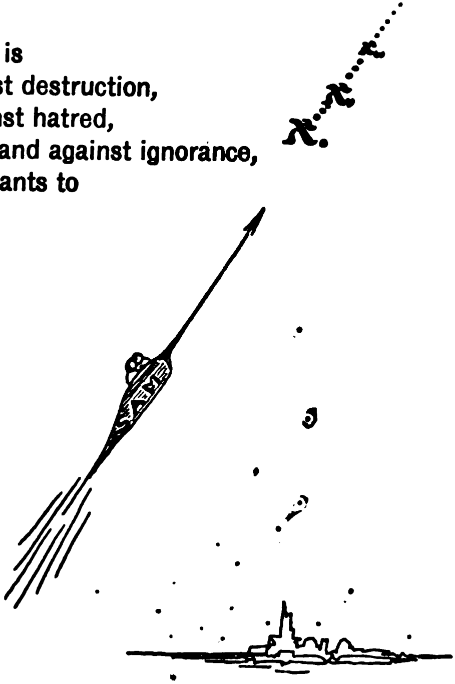
Dedication

This little book is
affectionately dedicated to
SAM,
who represents
the essence of what is BEST
in human nature —
the part of us which is
for LIFE, and against destruction,
for LOVE, and against hatred,
for KNOWLEDGE, and against ignorance,
the part of us that wants to
LIVE and LET live.



Preface

To those among you who
have read any of
my other little books on
mathematics,
so charmingly illustrated with
modernistic drawings by
my husband, —
Greetings again!
You of course will not be surprised
by this combination of
Art and Mathematics,
which go so beautifully together;
nor will you be startled by
the short lines,
for you know that
these are not intended to be verse,
not even "free verse",
but are used merely to
MAKE READING EASIER.
And, you will agree, that
any device which makes
the reading of mathematics
EASIER,
is welcome to many.

And, to any new readers, —
Greetings also.

To ALL
may I say that
as you get to know SAM better,
you will find him to be

a real FRIEND
who will bring you
HOPE
in your hour of need,
and a COMPANION
with whom you can share
your happiest moments.

Even if you have met him before,
in any of my other little books,
PLEASE do not say:
"Oh, I know him already
and therefore I can skip
Chapter 1 here."
For as I think more about him
and his characteristics become
more revealed,
I find him so HELPFUL
that I beg you to consider his
SIGNIFICANCE,
and I trust
with all my heart
that his
WISDOM and
VAST KNOWLEDGE and
LOVE and
INFINITE PATIENCE
may help you too,
as they have me.

Acknowledgments

It gives me great pleasure to thank Professor Garrett Birkhoff for having read the manuscript of this book and made some valuable suggestions. He is the Head of the Department of Mathematics at Harvard University and a great creative mathematician of international renown; it therefore gives me particular satisfaction to have his approval of this volume as well as of the whole enterprise of bringing to the general public some idea of the value and deep beauty of mathematics.

I should like also to give thanks to Stanley M. Rinehart, Jr., President of Rinehart & Company, Inc., and to their Editor-in-Chief, John Selby, who appreciated the need for such an enterprise and made possible its fulfillment. Their sympathetic understanding is a tribute to American publishing.

This venture, of course, involved the co-operation of many people in various departments — art, production, printing, et al. — to all of whom my thanks for their efficiency, patience and good will.

I wish also to acknowledge gratefully the kindness and exceptional courtesy of Mabel E. Jettinghoff, librarian in the Science Division of the New York Public Library, who made available to me any literature which I required.

L.R.L.

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Infinity

1 Our Good Friend, Sam

For those who have not met SAM before,
I wish to summarize
VERY BRIEFLY
what his old acquaintances
may already know,
and then to tell to all of you
MORE about him.

In the first place,
the name "SAM"
was first derived from
Science, Art, Mathematics;
but I now find
the following interpretation
much more helpful:
the "S" stands for
OUR CONTACT WITH THE OUTSIDE WORLD;
please note that
I do NOT say
that "S" represents "facts" or "reality",
for
the only knowledge we can have of
the outside world
is through our own senses or
"extended" senses —
like microscopes and telescopes et al
which help us to see better,
or radios, etc., which
help us to hear sounds
which we would otherwise
not be aware of at all,
and so on and so on.

But of course
there may be
many, many more things
in the world
which we do not yet perceive
either directly through our senses
or with the aid of
our wonderful inventions.
And so it would be
quite arrogant
to speak as if we knew
what the outside world "really" is.
That is why I wish to give to "S"
the more modest interpretation
and emphasize that
it represents merely
that PART of the OUTSIDE world
which we are able to contact, —
and therefore even "S" has
a "human" element in it.

Next:
the "A" in SAM represents
our INTUITION,
our emotions, —
loves, hates, fears, etc. —
and of course is also
a "human" element.

And the "M" represents
our ability to draw inferences,
and hence includes
mathematics, logic, "common sense",
and other ways in which
we mentally derive the "consequences"
before they hit us.
So the "M" too is

a "human" element.

Thus SAM is entirely human
though not an individual human being.

Furthermore,
a Scientist utilizes the SAM within him,
for he must make
"observations" ("S"),
he must use his "intuition" ("A")
to help him formulate
a good set of basic postulates,
from which his "reasoning powers" ("M")
will then help him to
derive conclusions
which in turn must again be
"tested" ("S" again!) to see
if they are "correct".

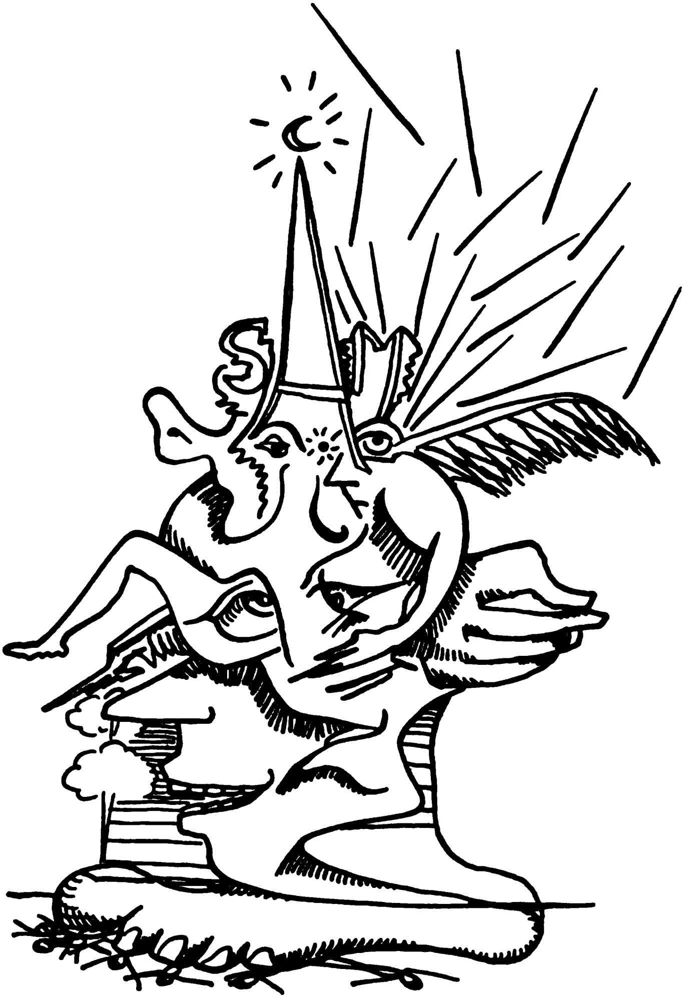
Perhaps you are thinking that
SAM and the Scientist
are really one and the same,
and that all I am doing is
to recommend that we all become
Scientists!
But you will soon see that
this is not the case at all.
For,
in the first place,
it too often happens, —
alas and alack! —
that when a Scientist is
not actually engaged in doing
his scientific work,
he may "slip" and not use
his "S", his "A", and his "M",
so carefully,

and will bear watching,
like the rest of us.
Of course when he IS doing
his scientific work,
he IS being watched by
his colleagues
who check and double-check
his "results",
so that
neither incompetence nor fakes
can get by —
at least not so often as they
otherwise might.
And he KNOWS that he is being watched,
and is therefore
VERY, VERY careful.
But when he is not doing
his scientific work,
he MAY get a little careless, —
though not necessarily so;
if the SAM in him is
really "in his bloodstream",
and not just his assistant in
the laboratory,
then he may indeed deserve
the title of SAMite, —
but not all scientists are like that,
by any means.
So, you see,
being a SAMite and being a Scientist
are NOT one and the same.

Besides,
a SAMite may not be a Scientist at all,
but an Artist!
For an Artist, too, must use
his "S" in order to "observe" the world,

his "A" ("intuition") to sense
some basic ways to translate his
"observations",
and his "M"
to derive his "results" in the form of
drawings, music, and so on.
Thus an Artist, too,
WHEN AT HIS BEST,
is a SAMite.

Now consider a person
who is SOMETIMES or OFTEN like this:
sAᴍ.
He is evidently relying very heavily on
his "intuition", his "hunches", his "emotions",
hardly checking to see whether
the "observations" of the outside world ("S")
and his own reasoning powers ("M")
show his "hunch" to be correct or not!
And so,
precious as our "intuition" may be,
it can go terribly "haywire"
if not checked and double-checked
by "S" and "M".
Thus, a person who
habitually behaves like this
is allowing his "S" and "M" to
become practically atrophied,
and is the wild, "over-emotional" type,
who is not only a nuisance to have around,
but is hurting himself and
not allowing himself to become
adjusted to the world he lives in.
Such a person,
with an exaggerated "A",
and atrophied "S" and "M",
has a feeling of "freedom",

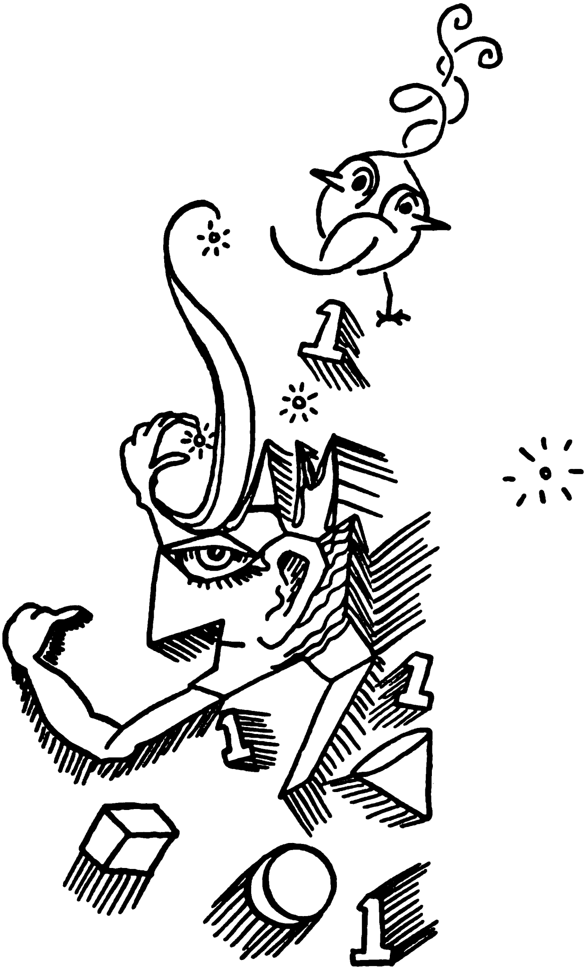


of not being held down by "S" and "M"
("facts" and "reason");
but, as you can easily see
this makes for Anarchy,
for a lack of "self-control" —
and can lead
to fatty degeneration from
feeling "free" to eat all he wants;
to the D.T.'s from
feeling "free" to drink all he wants;
to accidents because
he feels "free" to drive as fast as he wants
and to "hog" the road;
to a sadistic lack of
consideration for others
by feeling "free" to
kick them in the teeth for "nuttin' ";
to antisocial "black market" practices
from a similar feeling of "freedom",
giving "free" reign to the "A"
without the necessary consideration of
"facts" ("S") and "reason" ("M").
Needless to say this is a
PATHOLOGICAL FREEDOM
as against
a NORMAL, HEALTHY FREEDOM of
the well-balanced SAM
which is so necessary in society
in which EACH individual
must be guided by the SAM within himself
in order to avoid conflict with
the SAM in someone else.
This is something that
a bully does not understand —
that if he acts like a pathological sAM,
he induces sAMite-ism in others,
as in mob violence;

this is indeed a horrible "ism"
that can destroy a society as well as
individuals in it.

Similarly,
there is the SAM type:
he may be called the "tourist" type —
running around seeing this and that
but without the "imagination" ("A")
or the reasoning power ("M")
to put his observations together
with either heart ("A") or mind ("M"),
but is concerned only with
ISOLATED BITS OF INFORMATION:
he is like the man who,
seeing a crowd had gathered,
wanted to know what happened,
and, when someone told him
"Ein Mann hat sich dem Kopf zerbrochen"
(It happened to be in Germany),
corrected the speaker's grammar
and said "DEN Kopf!"
He knew his bit of grammar,
but what an inadequate reaction
under the circumstances,
don't you think?

And there is also the saM type —
one who can reason ("M")
but starts with perhaps
some postulate ("A") favoring murder.
Such a man would make
a wonderfully "rational"
homicidal maniac or crook
who could plan you a murder
calmly and rationally enough
to surprise any who are not familiar with



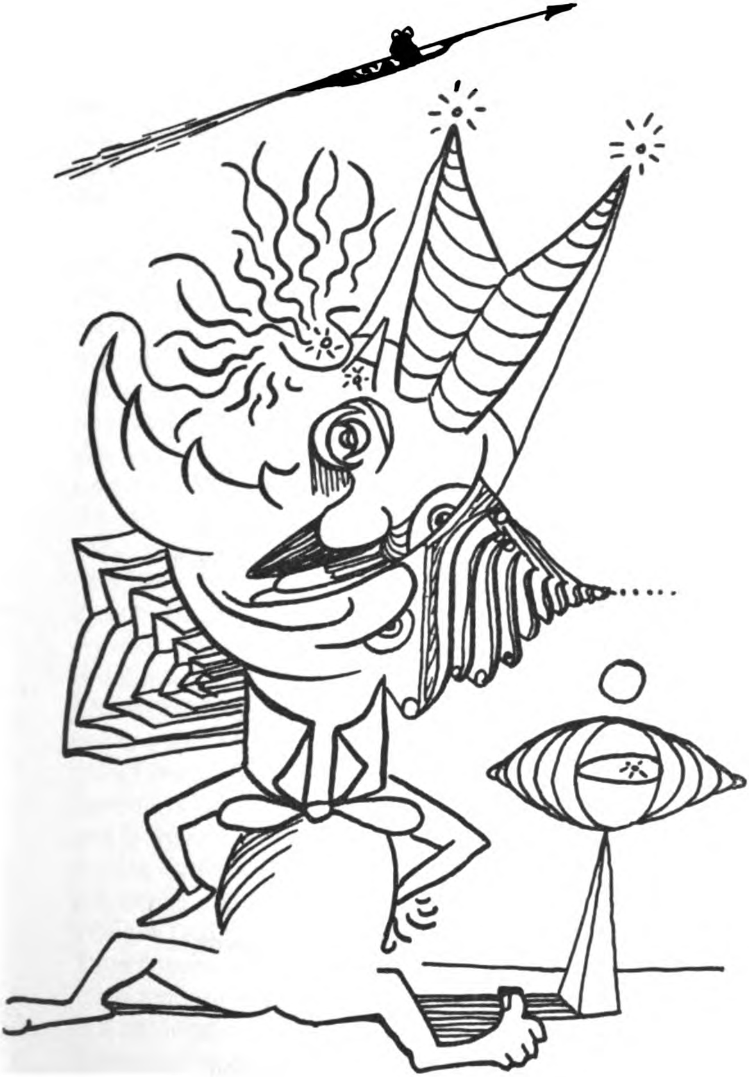
this saM type of pathological case.
And other abnormal conditions
(temporary or habitual)
might be represented by
SAm, or sAM, etc.
as you will see later (see p. 226).

Thus SAM gives us a way of
examining our own behavior
and that of others,
taking into account the "facts" ("S"),
and using imagination and sympathy ("A")
in a rational way ("M").

Are you perhaps thinking,
"Well, this may be interesting,
but
why all this talk about SAM,
when you are writing a book about
Infinity?"
To which the answer is:
The yearning for Infinity,
for Immortality,
is an "intuitive" yearning ("A"):
we look for support for it
in the physical world ("S"),
we try to reason about it ("M"),—
but only when we turn
the full light of SAM upon it
are we able to make
genuine progress in considering
Infinity.

As you will see.

There is only one more point
I must make here:



Namely, that
even being a well-balanced
SAMite —
and not a pathological anti-SAMite
like sAM, etc. etc. —
is NECESSARY but NOT SUFFICIENT.
You will probably agree that
it is further necessary
to have our SAM up-to-date.
For he is a GROWING boy,
and what was good enough for him in 1800
is utterly inadequate in 1953;
and unless the "S" is up-to-date
and the postulates ("A")
and reasoning ("M")
are appropriately MODERN (see p. 335),
we cannot make proper
ADJUSTMENT in the world TODAY.
And ADJUSTMENT is what we must have.
For adjustment means
SURVIVAL,
and that is a MINIMUM demand —
for, without survival
we need not bother to study anything
we just won't be here to tell the tale.

And so let me summarize
by saying that the
ANTI-SAMITES
hurt not only themselves,
by getting "ulcers", nervous breakdowns,
drinking excessively, etc. etc.,
but hurt others also,
for from their ranks are recruited
those who advocate war and destruction,
the homicidal maniacs, the greedy crooks,
the gamblers, the drunken drivers,

the liars, et al.
If I have not yet
made it sufficiently clear
that all such are
pathological ANTI-SAM-ites,
it will become clear, I hope,
by the time you have
finished reading this book.

Just a word more about
FREEDOM —
you have seen above (p. 9)
the pathological idea of freedom,
but when you consider this important concept
from SAM's WELL-BALANCED viewpoint,
you will see that,
from this point of view,
the "feeling" of freedom ("A"),
being supported on one side by "S"
(the "facts" of the outside world),
and on the other by "M"
("sweet reasonableness") —
is definitely NOT the
ANARCHICAL freedom of sAm,
but is a sort of
CONTROLLED FREEDOM —
controlled by facts and reason
and is therefore SELF-controlled
(by the SAM within us)
and hence implies
VOLUNTARY COOPERATION rather than FORCE.
Thus anyone who demands
"freedom unlimited" as his right,
is a pathological sAm,
a destructive creature;
whereas,
in mathematics

you will find the
CONTROLLED FREEDOM of SAM
and you will feel refreshed to see
how genuine progress can be made
with this kind of freedom.

2 Infinity in the Physical World

Of course you know that
the Infinite
is a subject which
has always been of the deepest interest
to all people —
to the religious,
to poets,
to philosophers,
to mathematicians,
as well as to
T.C.Mits
(The Celebrated Man-in-the-Street)
and to his mate,
Wits
(the Woman-in-the-Street).
And it probably interests you,
or you would not be reading this book.

Now some people
make the MISTAKE of thinking that
Infinity is merely
something VERY LARGE!
But of course
what is "VERY LARGE" to one person
may seem quite small to another.
Thus,
there are some peoples,
untrained in these matters,
in whose very language
there is no word for

a number greater than 2 (or 3) —
after that they merely say "many"* —
one, two, many!
And perhaps they think that 3 is
Infinity!

Others think that
the number of stars is infinite —
and yet
astronomers tell us that
one can see only
a finite number of stars,
even with the most powerful telescopes:
a very large number of course,
but not an Infinity of them.
Sir Arthur Eddington,
a well-known British astronomer,
gives the following
"celestial multiplication table":
A hundred thousand million stars
make one Galaxy;
a hundred thousand million galaxies
make one Universe.‡

Still others believe that
the number of grains of sand on a beach
is infinite —
and yet if one estimates
the number of grains in a small sample,
and then estimates the dimensions of the beach,
the TOTAL number of grains of sand

---

*For a delightful discussion of these
and many other illustrations, see:
T. Danzig: Number, the Language of Science.
E. Kasner and J. R. Newman: Mathematics and the Imagination.
G. Gamow: One, two, three, ..., Infinity.

‡See p. 4 of "The Expanding Universe" by
Sir Arthur Eddington (1946).

on any beach
turns out to be again a LARGE number
but NOT INFINITY!
(See the books referred to in
the footnote on p. 18).

I hope these examples
are not beginning to bore you —
if you will bear with them
a little longer,
you will realize that
no number, however large, is
Infinity!*
Consider, for instance,
the next two examples:
you doubtless know that
any material substance,
no matter how smooth it appears,
is supposed to be made up of
"molecules",
and that the molecules are made up of
"atoms",
and that the atoms are made up of
"electrons" —
for all this there is
the best possible scientific evidence,
as witness
"atomic energy",

---

*Later you will see that,
in mathematics,
there are various kinds of infinities,
and that they have been named
"transfinite numbers" —
but of course these are
NOT ORDINARY NUMBERS,
and require a set of postulates which
is ENTIRELY DIFFERENT from
the set of postulates for
ordinary numbers.
But let us not get ahead of
the story!

which is obtained by
actually "splitting" atoms
(generally called "fission" of atoms).
Now,
chemists tell us that
one cubic centimeter* of any gas
contains approximately
(under certain conditions of temperature and pressure)
27 million million million MOLECULES,
which may be written

$$27 \times 10^{18}.‡$$

And,
since electrons are
even smaller than molecules,
there is a still greater number
of ELECTRONS in
this small quantity (1c.c.) of a gas.
Now, believe it or not,
scientists have even dared to ask
"How many electrons are there
in the entire UNIVERSE?" —
that would include
the Earth,
the Sun,
the entire Solar System,
all the Stars in
all the Galaxies of stars!
You must admit that
this is a stunning question,
and deserves a stunning answer!
Well,

---

*There are approximately
2.4 centimeters to one inch.

‡10^{18} means
1 with 18 zeros after it,
just as 10^2 means
1 with 2 zeros after it,
or 100.

Sir Arthur Eddington,*
the astronomer referred to above,
estimated this number
and found that
not only is it FINITE
but it is not nearly as large as
your "hunch" might lead you to believe!
Indeed he estimated it to be
of the order of magnitude of

$$1.29 \times 10^{87}.^{*}$$

This is of course
a VERY large number,
but it is
NOT INFINITY
by any means;
indeed, mathematicians,
and you too, for that matter,
can write numbers MUCH larger than this,
as, for example, 10^{100},
now popularly called a "googol",
a name invented by the young nephew of
one of our great American mathematicians,
Edward Kasner,
formerly Adrain Professor of Mathematics
at Columbia University,
and now Emeritus.

And who can prevent anyone
from writing still larger numbers,
like 10^{1000}, etc., etc., etc.?
The IMPORTANT thing to know is that
NO number, HOWEVER LARGE, is
Infinity!‡

*See p. 68 of his book to which
I referred above (on p. 18).

‡But see the footnote on p. 19.

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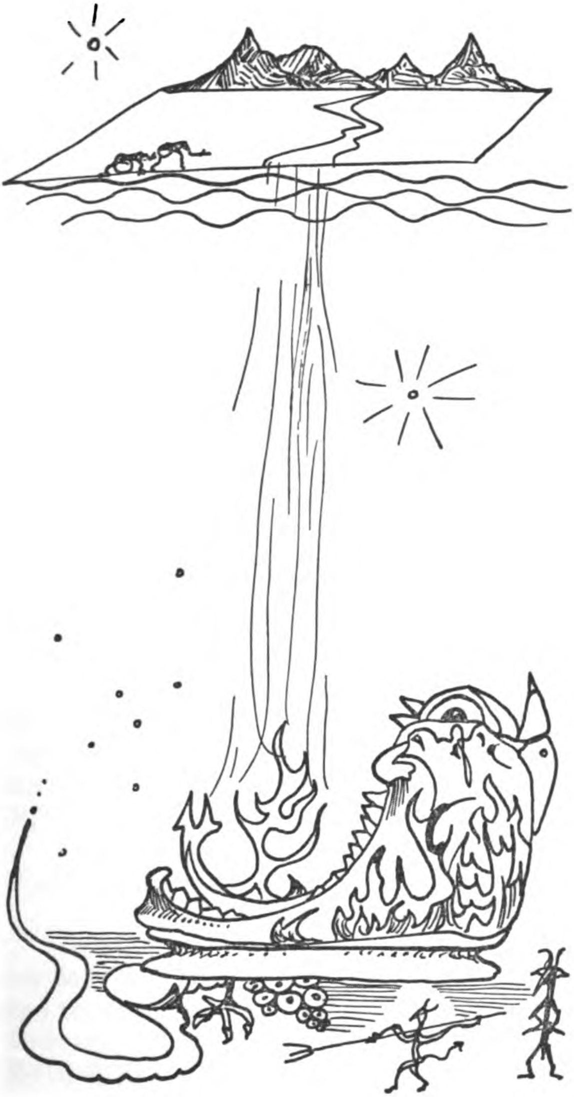
It may be interesting to note that
a student who knows
a little chemistry,
enough to have heard that
there are 27×10^{18} molecules in
1 c.c. of any gas,*
(under certain conditions of
temperature and pressure),
such a student often has a "hunch"
that 1.29×10^{87} is not nearly
large enough
as an estimate for
the number of electrons in
the entire universe!
His difficulty probably is that
he does not realize
the tremendous POWER of an
EXPONENT —
he probably does not "sense"
how VERY much bigger
1.29×10^{87} is than 27×10^{18}.
Thus the "hunch" of
a person with some education
may still be "off the beam" —
all "hunches" must be
checked and double-checked
no matter what their source!
And I recommend that anyone who
doubts Eddington's estimate
had better read his argument
(in the reference given above)
rather than follow his own unsupported
"hunch".

A final illustration:
You know that

*See p. 20.



when people thought that
the earth was flat, they naturally
wondered whether it goes on
to Infinity in all directions,
like a Euclidean plane,
OR
does it have an edge, a boundary,
and
when you reach the edge,
what happens then?
Do you fall off into Hell
or what?
Well,
as you know perfectly well,
neither alternative was inevitable,
for,
as it turned out,
the earth is spherical,
and therefore has no "boundary",
and is thus
UNBOUNDED though FINITE!

Similarly, now,
in thinking about our
three-dimensional space,
we again naturally wonder whether
it goes on to Infinity
in all directions
OR
does it have a boundary?
And, if so,
what is "beyond" the boundary?
Well, again, there is
excellent scientific evidence*

---

*See "The Meaning of Relativity"
by Albert Einstein himself,
published by the Princeton University Press in 1950.

that
our Universe itself is
UNBOUNDED but FINITE!

And thus we see that,
so far as
the physical world is concerned,
we have not found anything which is
infinitely large
(not even
the physical universe itself,
nor
the number of electrons in it!);
nor anything which is
infinitely small ("infinitesimal"),
but instead
things are made up of
minute but FINITE particles,
like electrons and others,
in the case of matter,
and like "quanta"
in the case of energy.

But is then
the human spirit defeated in
its search for the
INFINITE?

Not at all!

For,
not only do
religious people continue to
believe in an
INFINITE GOD,
and in
IMMORTALITY,

but even those
who are not religious
find other ways of
expressing this yearning
by thinking of themselves as
a part of an infinite chain of Life.
And,
as you will see
in the rest of this little book,
the Mathematicians, too,
have NOT abandoned
their study of Infinity:
they have been interested in it
for many centuries,
and have,
in their own inimitable way,
developed it RECENTLY into
a most POWERFUL weapon of thought,
not only interesting to
mathematicians and philosophers,
but even of the
utmost practical use besides!

3 Infinity in Mathematics

And so,
let us now look at
what the mathematicians have done with
Infinity.

In the first place,
they distinguish between
"potential" and "actual" Infinity,
a distinction which will soon
become quite clear to you.

Take first
"potential" Infinity,
which can be illustrated by
a few examples from
quite elementary mathematics:

(1) An illustration from
ordinary arithmetic or
ordinary algebra:
you will of course agree to
the correctness of the answers
in the following little
division examples:

12/12 = 1; 12/6 = 2; 12/4 = 3;
12/3 = 4; 12/2 = 6; 12/1 = 12;
12/0.1 = 120; 12/0.01 = 1200;
12/0.001 = 12000; etc., etc.

Please note that
I have chosen the same
numerator, 12,

in all cases,
but
the DENOMINATOR is DECREASING
all the time,
and hence
the ANSWER gets larger and larger.

And of course,
as the denominator continues to
DECREASE by moving
the decimal point further to the left,
and thus APPROACHES ZERO,
the ANSWER will continue to
INCREASE,
growing larger and larger
WITHOUT END, —
and we therefore say that
the answer
APPROACHES INFINITY.
You can easily see that
by this process of
continuing to move
the decimal point in the denominator
more and more to the LEFT,
the denominator will
NEVER become zero,
and the answer will
NEVER become infinity,
and that is why it is
appropriate to call this answer
"potential" but NOT "actual" infinity.
The symbol for this
"potential" infinity is ∞,
and is of course
not a number at all, —
it is beyond any number,
it is "off the map".

Hence
if you write $12/0 = \infty$
where the denominator is
ACTUALLY zero
you are not really
performing a division and
getting a "legitimate" answer,
since ∞ is not a number,
and you are no longer
"playing the game" of
arithmetic or algebra,
in which NUMBERS constitute
the "equipment";
and consequently
12/0 is a "foul" here and is therefore
NOT ALLOWED.
And so
the only meaning ∞ has
is
"potential" infinity
which may be
APPROACHED but NEVER REACHED.
Occasionally a young student will say:
"But 12/0 says that
you are dividing 12 by nothing
and hence
you are not dividing 12 by anything
and hence
the answer should be 12."
I mention this only because
it illustrates how
a poor "translation" from
the language of mathematics into
some other language, say English,
results in a
MISUNDERSTANDING.
If you play with words,

as this young student did,
it is quite easy to become confused,
merely because
the original meaning can thus
become distorted.
Thus 12/0 is NOT INTENDED to mean
"you are not dividing 12 by anything",
but its meaning is,
in mathematics,
12 divided by a number which is
so small that
it has no magnitude at all,
and therefore
the answer will be so large that
it is no longer a number at all,
and we call it "potential" infinity
and designate it by the symbol ∞.
And of course the same would be the case
if a number other than 12
were taken for the numerator,
say 10 or 100 or 53 or
any other number, a,
EXCEPT zero,
for if BOTH numerator and denominator
are zero,
this turns out to be
a particularly interesting and
practical case
(See Ch. 18).
And so,
whenever the basic rules of
arithmetic or algebra
are given,
one must never fail to say that
"division by zero" is ruled out,
that is,
a/0 (where a itself is not zero)

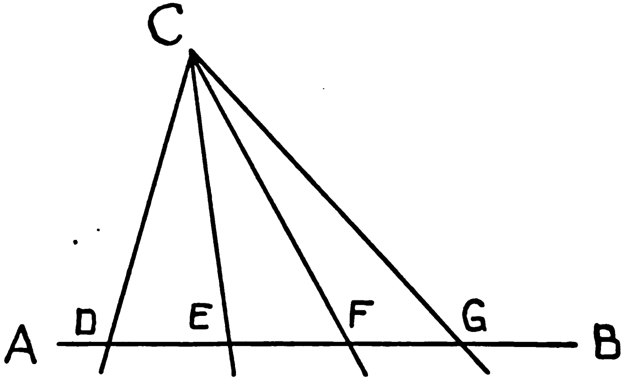
is a "foul"
in the "games" of
ordinary arithmetic and algebra.

So far then
you are probably not impressed
by this ∞,
which seems
too big to be of any use,
and too far away for comfort.
But may I point out that
even so
it has the advantage that
a mathematician is
always AHEAD of
any PRACTICAL MAN,
for ∞ puts no limit on
the size of numbers —
so that
as the budget grows from
millions to billions to trillions etc.,
or as the physicist needs
a number big enough to estimate
how many electrons there are
in the entire universe,
or how big is the entire universe,
and so on and so on,
the mathematician has left the door
wide open to PLENTY
and then some,
with his ∞.
But this is far from being
the whole story of Infinity in
mathematics, —
it is only the BEGINNING of
the mathematician's "games" with
Infinity.

Take next:

(2) an illustration from
elementary Euclidean Geometry:
here a straight line
is supposed to be of
infinite length,
meaning that
the line-segment AB

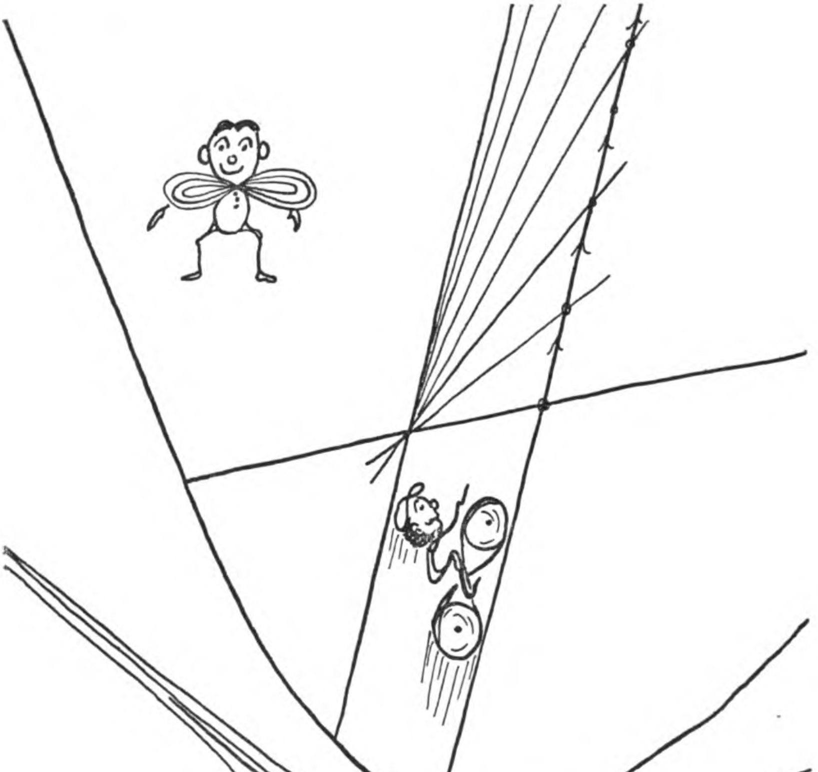
may be prolonged
in either direction,
to the right or to the left
without limit—
and is thus another illustration of
"potential" infinity.



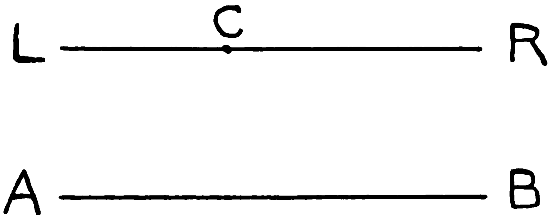
And now,
if you have a point, C,
outside the line AB
you can of course draw

many straight lines through C
which will intersect the
given line
at D or E or F or G
and so on and so on,
as far to the right or
as far to the left
as you please.

Now,
as the point of intersection
goes farther and farther to the right,
it will APPROACH INFINITY,
and
to say that a straight line through C
"meets" the given straight line
at Infinity
(or, as they say in mathematics,
at an "ideal" point)
is only another way of saying
that there is
no point of intersection,
or that
the two straight lines are then
"parallel".
Similarly,
if the straight line through C
is so drawn that
it intersects the given line
farther and farther to the LEFT
the same thing will happen —
namely,
the point of intersection will go
farther and farther to the LEFT,
and will APPROACH INFINITY
on the LEFT,
until again the lines are parallel.



Now,
in Euclidean Geometry,
there is a BASIC ASSUMPTION that
through a given point (C) which is
NOT on a given straight line,
there is ONE and ONLY ONE
straight line which is
parallel to the given straight line, —
so that the one extending
to the RIGHT of C,
and the one extending
to the LEFT of C
are merely parts of
one and the same straight line,
thus:



If CR is parallel to AB and
CL is parallel to AB,
then
LCR is one and the same straight line.
All this is simple enough.
But let us look at
one of the implications of it:

You have agreed that
if LR and AB are parallel,
then they will
"meet" at an "ideal" point
way off to the right,

and at an "ideal" point
way off to the left,
and therefore
LCR and AB
(both prolonged of course)
seem to have TWO "ideal" points in common;
BUT
there is a postulate in
Euclidean Geometry
which says that
two straight lines
cannot have more than one point
in common.
Consequently,
if you are going to apply
the word "point"
to these "ideal points",
(as is done in
what is known as
Projective Geometry*),
then you would be breaking
one of the basic rules (or postulates)
of your game
which demands
not more than
ONE point in COMMON
for any two straight lines.
Now what would you do
if you were faced with
this dilemma?
Would you throw out
the "ideal points"
and lose all the advantages of
Projective Geometry?*
Or would you be brave enough

---

*See "Projective Geometry" by
Veblen and Young.

to go along with
the mathematicians
who merely consider
the "ideal point" which is
way off to the right
AND
the "ideal point" which is
way off to the left
as being
ONE AND THE SAME POINT?

Does this strike you as being
TOO FANCIFUL?
TOO IMPRACTICAL?
in fact a little "balmy"?
Well,
it is indeed fanciful,
but NOT impractical,
since, as I said above,
it made possible
the development of
a new and useful
branch of mathematics:
Projective Geometry!

Thus you see,
although this is only the beginning
of the story about
Infinity in mathematics,
that
the mathematician has already
scored twice
over the "PRACTICAL" man:

(1) he has an
ABUNDANCE
which is ready to cope with

any quantitative need
in which the "practical" man
might find himself
and

(2) by not being afraid of
FANTASY,
he is able to
develop new tools of thought
which the "practical" man
would never have dared
to think of,
but which
he gladly uses
when they are handed to him!

Just a word about the FANTASY:
as you see,
the mathematician feels quite at home
in fantasy —
he demands only that
there shall be
NO CONTRADICTIONS —
you cannot have
a basic rule and then
put in another rule which
contradicts it!
And, you will admit,
this is a most reasonable demand —
for, in any game,
if you had two rules which
contradicted each other,
what would the poor umpire do?
He has troubles enough as it is!

So,
barring contradictions,

the brave mathematician
makes PROGRESS through FANTASY,
while the LESS IMAGINATIVE
"PRACTICAL" MAN
at first reluctantly
but then inevitably and gladly
picks up the beautiful ideas —
and then thinks he is the "big shot"!
The worst of it is that
he never even says "Thank you" —
and this is a vast understatement.*

Indeed we shall be lucky if
some of these "practical" men
do not destroy us all
by their ABUSE
of the wonderful idea of
atomic energy, for example.

We really should reconsider
our use of the term "practical"!
Is it really "practical"
to abuse a beautiful idea
and turn it AGAINST
the human race?
Are not these destroyers
the most IMPRACTICAL of men?

*Read "Men of Mathematics" by
E. T. Bell
if you wish to know
the human side of
the lives of
many great mathematicians!

4 More About "Potential" Infinity

Consider a cone
whose base is a circle
having center A,
and whose axis, AB,
(i.e. the line joining
the center of the base, A,
to the vertex of the cone, B)
is perpendicular to the base.
Such a cone is called
a right circular cone
("right" because of the right angles
formed by the axis at A,
since AB is perpendicular to the base,
and "circular" because
the base is a circle).
Now, you can cut such a cone
in various ways:

(1) by a cut which is
perpendicular to AB
at any point of AB:
such a "section" will always be
a CIRCLE,
though it will vary in size,
getting smaller as you come
nearer to B,
as shown in Fig. 1 on p. 41.
(Of course if you cut
right through B,
the circle will have

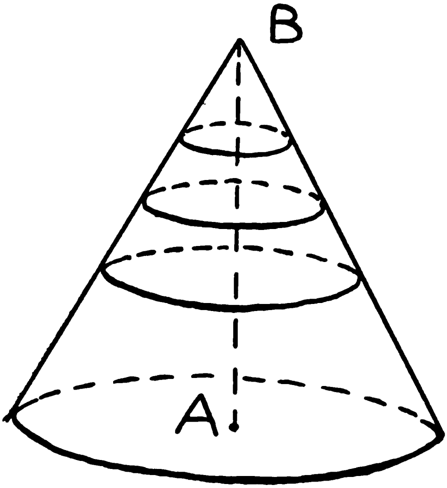


Figure 1

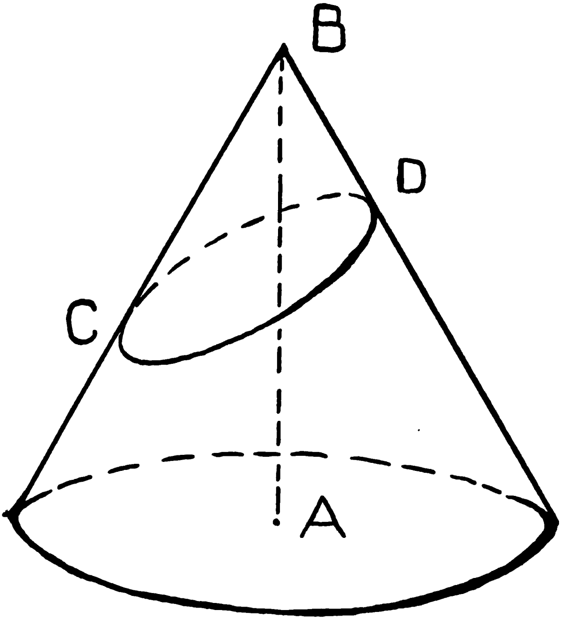
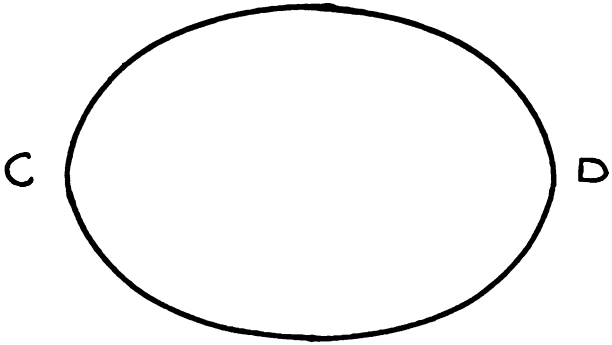


Figure 2

dwindled down to a point
or a "circle" of zero radius;
in that case
it is called
a "degenerate" circle —
no offense intended.)

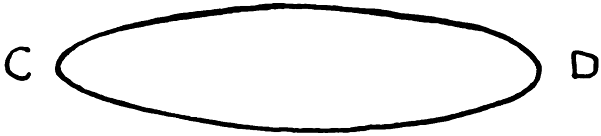
(2) by a cut which is
slightly tipped,
i.e. NOT perpendicular to AB,
as in Figure 2 on p. 41.

Such a "section", CD,
is an ELLIPSE,
and its shape,
when not looked at in perspective
(as in Fig. 2),
looks like this:



And of course,
as you tip the cut more and more,
the ellipse changes in shape,
becoming relatively

longer and narrower,
like this:



(And again,
if you do this at
various points of AB,
the ellipse will become
smaller as you get nearer to B,
until finally,
right through B,
you get a mere point again
which you may now regard as
a "degenerate" ellipse.)

The ellipse, as you probably know,
is a very important curve,
being, among other things,
the path in which
a planet goes around the sun.*

Now imagine that the base of the cone
is pushed lower and lower
(without changing the position of B)
and APPROACHES INFINITY,
so that the cone is now
"open" at the bottom, like this:

---

*According to the Einstein Theory of Relativity
the path of a planet is
NOT EXACTLY an ellipse
a fact especially noticeable in
the case of the planet Mercury.
See my little book
"The Einstein Theory of
Relativity".

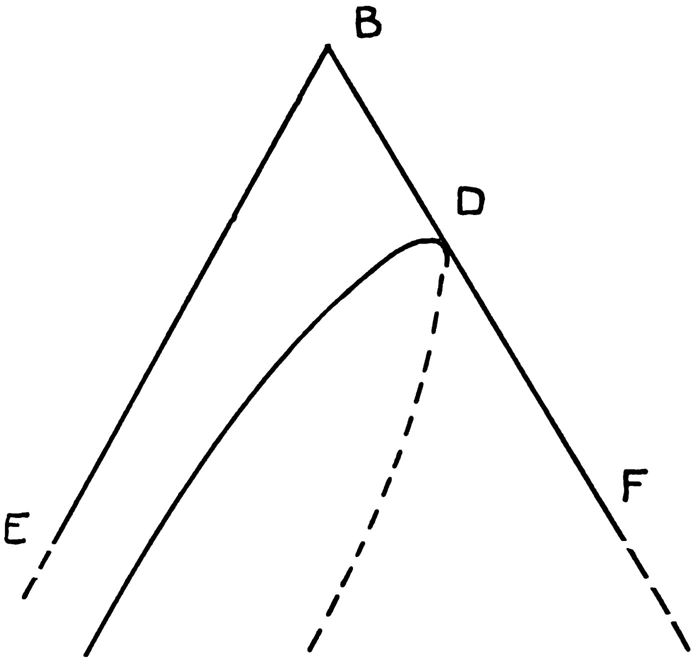
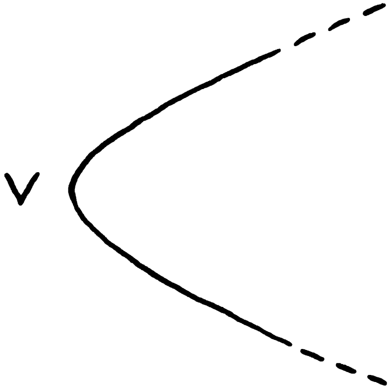


Figure 3

the dots in Fig. 3 indicate that
the lines BE and BF may be prolonged
as far as you please.
And now suppose you cut this
infinite cone (Fig. 3)
by a plane which is tipped
exactly enough to be
parallel to BE.
In this case:

(3) the "section" cuts BF at
some point, D,
but does not cut
BE (even when prolonged) at all,

and the section now
is not a "closed" curve any more
(as were the circle and ellipse).
This section is called
a PARABOLA,
and its shape,
when not seen in perspective
(as in Fig. 3),
is:



or

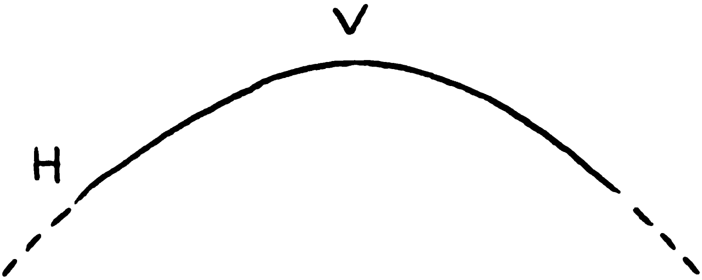


Figure 4

Just where the "vertex", V, of
the parabola
is located

depends of course on
which way you are holding the cone.
Incidentally,
as you probably know,
the parabola in Fig. 4
shows the path of a projectile
(or a baseball)
shot up (or thrown up) at
an angle at H.*

And finally
suppose you now have
a DOUBLE cone,
extending to infinity in
BOTH directions,
as in Figure 5 on p. 47;

and

(4) suppose you now tip the cut
still more,
so that it is not even parallel
to BE,
but will cut BF somewhere,
say at D,
and BG at K,
so that your section now
consists of TWO parts,

---

*The path of a projectile is
a parabola ONLY under
the ideal condition of
no air resistance;
otherwise,
under actual physical conditions,
it is not exactly a parabola,
and is studied in "Ballistics" —
but this study of actual projectiles
does not affect
the mathematical sections of a cone
discussed above.

one open branch on the
lower part of the cone (from D)
and the other open branch
on the upper part of
the cone (from K).

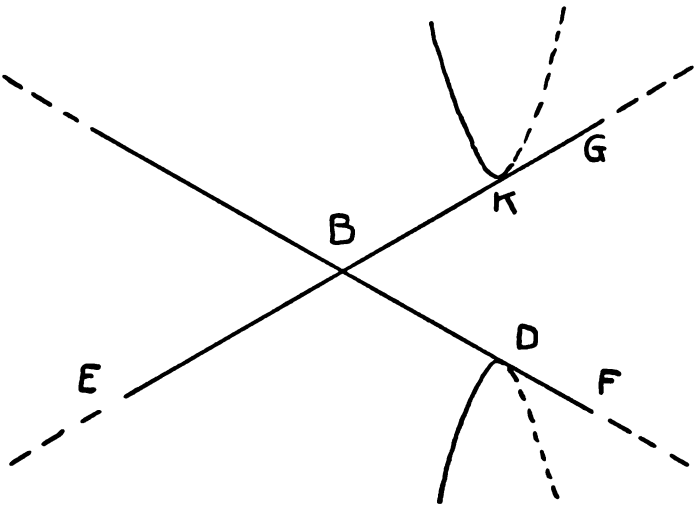


Figure 5

Both branches together
are considered to be
a single curve called
a HYPERBOLA.
And though each branch of
a hyperbola
may seem to you to have
the same shape as that of
a parabola,
it is in fact QUITE different,
as you will presently see.*

---

*See the footnote on p. 59.

(And of course here too
the cut through D and K
may be made nearer to B,
and when it goes right through B,
the section becomes merely
a pair of
intersecting straight lines,
like this:

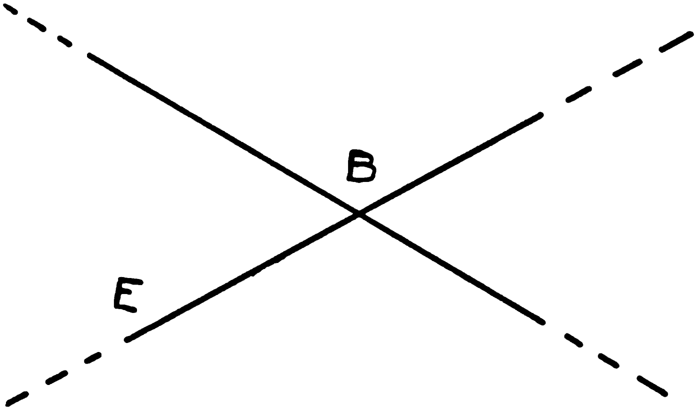


Figure 6

and may be regarded as
a "degenerate" hyperbola.)

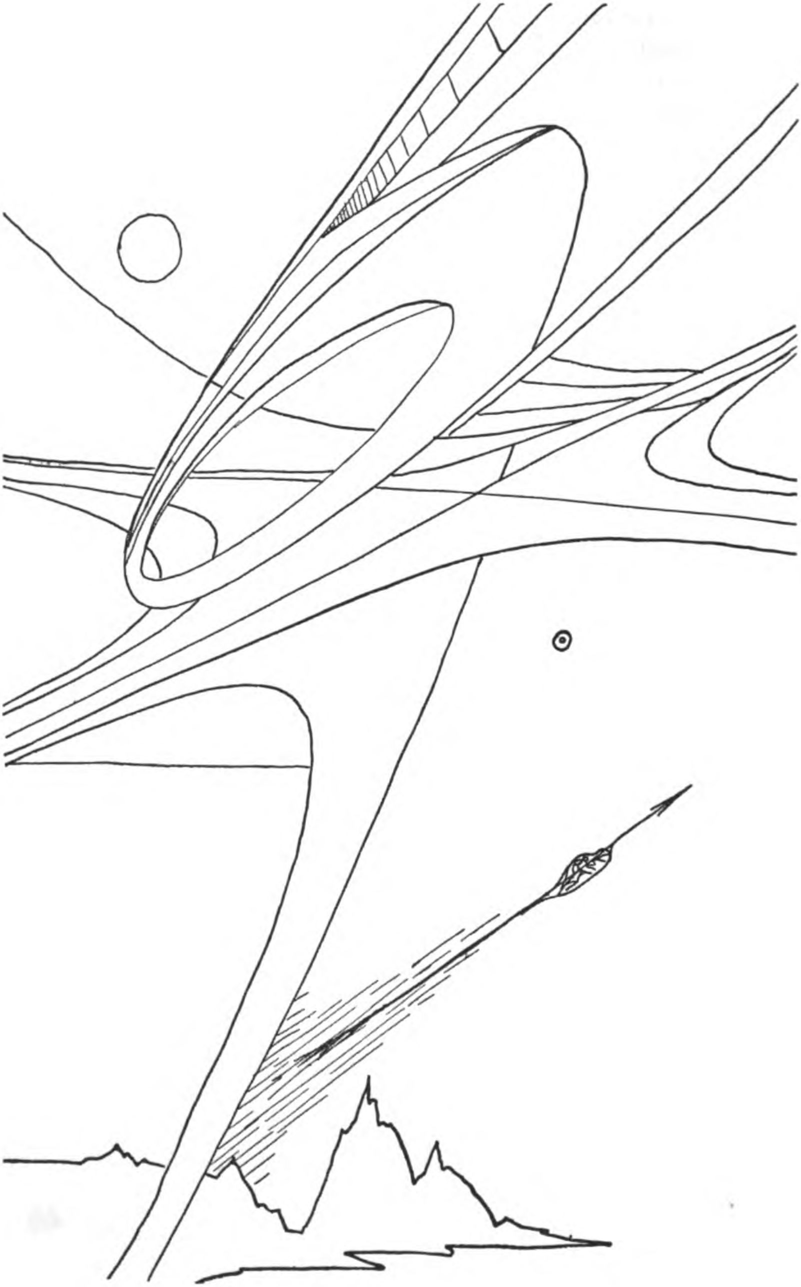
If you had used
a double cone
from the start,
you can easily see that
the first three kinds of sections,
the CIRCLE, the ELLIPSE, the PARABOLA,
would not have been affected by
the presence of
the upper extension of the cone;
for even the section which produced

the parabola
would just miss the upper cone,
being parallel to BE.
Thus you see that
a cone can be cut in various ways
by a plane,
the resulting sections being
therefore
known as "CONIC SECTIONS",
and are limited to being
a CIRCLE,
an ELLIPSE,
a PARABOLA,
a HYPERBOLA,
and a few "degenerate" forms of these,
as mentioned above.

These conic sections were studied
long, long ago
by the Greeks,*
several centuries B.C.,
though they did not suspect
at that time
that they would have
important applications to
astronomy, ballistics, etc.,
applications which came
many centuries later —
another illustration of a case
in which
the mathematician's curiosity
was FAR AHEAD of
any PRACTICAL use
and thus his ideas were READY
long before they were needed.

---

*Look up the work of APPOLONIUS
a celebrated Greek mathematician.



So "hats off" to mathematicians,
please!

And then,
about the middle of
the seventeenth century,
a great French mathematician,
Descartes,
conceived the amazing idea of
putting Algebra and Geometry
together,
a wonderful idea which again
bore much fruit.
I want to tell you
VERY BRIEFLY
about this,
because
we shall find it useful in
our further discussion about
Infinity.

Of course you know
what a "graph" is,
and you know that
one way of making a graph is
to draw
two perpendicular axes,
let us call them the
X-axis and the Y-axis,
intersecting at the "origin", O,
as shown in
Fig. 7,
on p. 52:

And now,
any point in the entire plane
may be represented by

a pair of numbers, thus (3, 2),
the first of which, 3, tells
how far you must go
along the X-axis,
and the second number, 2,

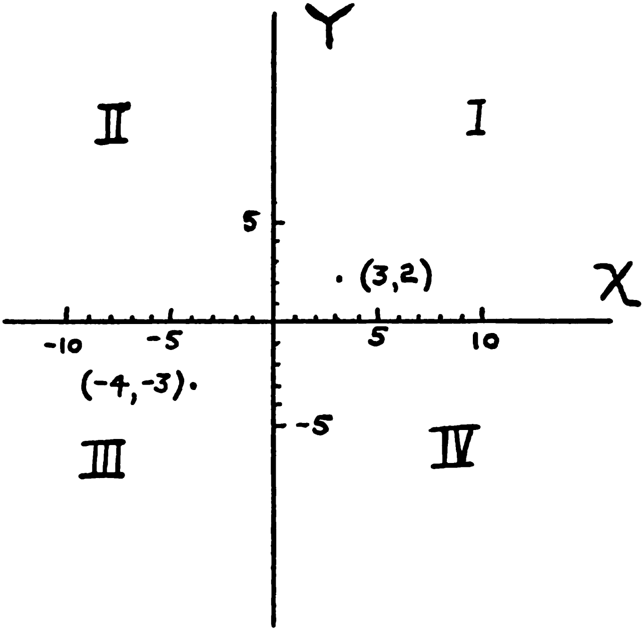


Figure 7

tells how far you must go
parallel to the Y-axis,
in order to arrive at
the point in question,
as shown in Fig. 7.
And, if the first number is negative,
you must go
to the LEFT on the X-axis,
and

if the second number is negative,
you go DOWN on the Y-axis,
as shown, also in Fig. 7.
The four "quadrants"
formed by the two axes
are numbered in
a counter-clockwise direction,
as illustrated in Fig. 7,
so that
(3, 2) is said to be in
the FIRST quadrant,
(−4, −3) in the THIRD quadrant;
and, similarly,
(−1, 4) is in the SECOND and
(5, −3) in the FOURTH quadrant,
etc.
And, obviously,
(5, 0) is right ON the X-axis,
(0, 3) is ON the Y-axis,
and (0, 0) is at the ORIGIN itself.

Now, with this simple device,
it is possible to
translate every algebraic equation
containing two unknowns, x and y,
into a graph;
thus if the equation is:

$$x + y = 10 \tag{1}$$

then any pair of numbers whose
sum is 10
will "satisfy" this equation;
for instance, take
$x = 7$ and $y = 3$.
By placing the point (7, 3)
on the graph,
as shown in Fig. 8,

we have a geometric representation,
by means of a point,
of a PAIR of numerical values of
x and y which
satisfies the given equation.
Similarly,
we could take x = 4, y = 6
and get another such point,
etc.:

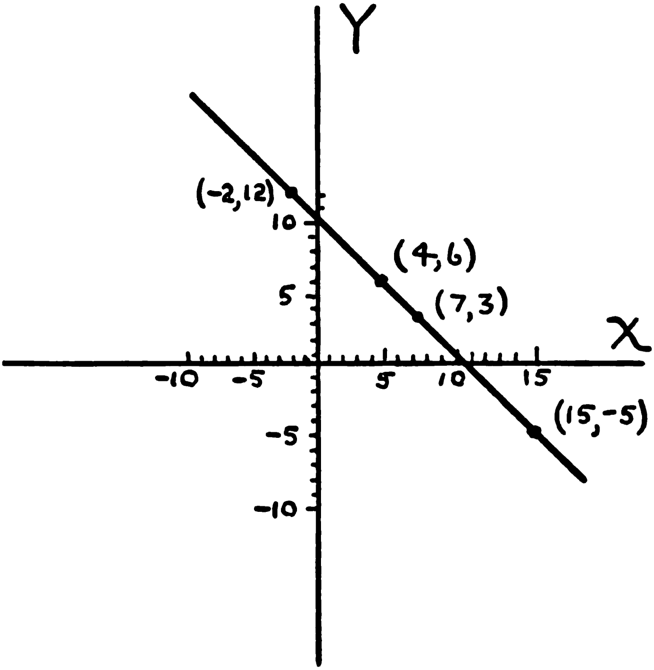


Figure 8

And it can easily be shown
that all points which
satisfy equation (1)
lie on a STRAIGHT LINE,

and, conversely,
every point on this line
will SATISFY equation (1)
so that
this straight line may be regarded
as a "portrait" of equation (1).
And of course
the line may be prolonged
as far as you please
in either direction,
thus going into the
second and fourth quadrants,
going through points like
$(-2, 12)$ and $(15, -5)$, etc.,
which still satisfy equation (1).

Now it turns out that
all the CONIC SECTIONS
may be represented by
equations of the SECOND DEGREE,
that is,
equations which have
at least one term of the second degree
and none of any higher degree.*

For instance:

$$x^2 + y^2 = 25 \tag{2}$$

is of the second degree

---

*Thus $5x^2$ is a term of the
second degree
since the exponent of x is 2
(the coefficient 5 has nothing to do
with the "degree" of the term).
The term $7xy$ is of the
first degree in x, and of the
first degree in y,
and is thus of the SECOND degree in
BOTH x and y,
etc.

and represents a circle
of radius (r) equal to 5
and center at O:

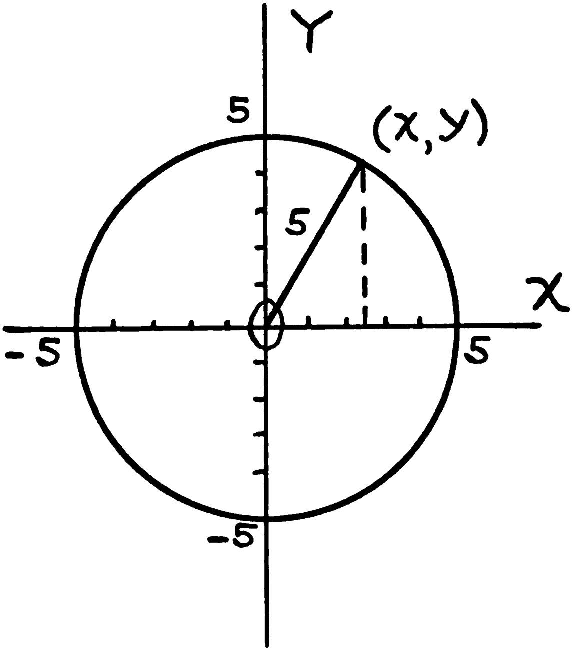


Figure 9

since any point (x, y) on the circle
satisfies equation (2)
by the well-known
Pythagorean Theorem,
as seen in Fig. 9.
Similarly

$$xy = 9^* \qquad (3)$$

---

*This is an equation of the
SECOND degree,
as explained in
the footnote on p. 55.

is represented graphically by
a hyperbola,
like this:

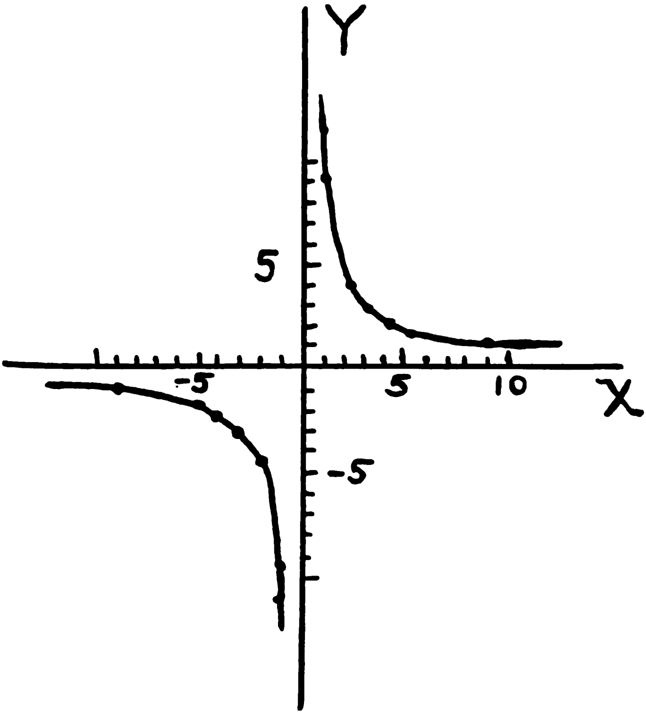


Figure 10

a curve which you can easily
trace out for yourself
with the aid of the following table (p. 58)
of pairs of values for x and y
which "satisfy" equation (3):

All that equation (3) demands is
that the PRODUCT of

x	y
1	9
2	9/2
3	3
4	9/4
5	9/5
9	1
−1	−9
−2	−9/2
−3	−3
−4	−9/4
−5	−9/5
−9	−1
.	.
.	.
.	.
.	.
etc.	etc.

any value of x and
its corresponding value of y
shall be equal to 9;
and this is true of
any pair of values of x and y in
the above table,
as you can easily check.
And of course you can find
many, many more such pairs, —
as many as you please.
Note that
the smaller the value of x
which you choose,
the larger must be the value of y
in order that their product
may remain 9, as required by
equation (3).

Thus the nearer you approach
the origin, from the right,
along the X-axis,
the smaller is the value of x
and hence
the larger the value of y —
in other words,
as x approaches 0
y approaches ∞,
so that
the curve will NOT cross the Y-axis
at any FINITE distance from
the origin, O.
Or we can say (as in Chapter 3) that
the curve will "meet" the Y-axis
at an "ideal point";*
similarly,
the larger the value of x,
the smaller must be the value of y,
to keep the product down to 9,
and hence,
as x approaches ∞,
y approaches 0,
and the curve will "meet" the X-axis
at an "ideal point" way off to the right.
Note also that
since the product of
two negative numbers

---

*Another way of expressing this
is to say that the curve
approaches the Y-axis "asymptotically",
or, the Y-axis is an
"asymptote" of the curve.
And, similarly, the X-axis is also
an asymptote of the curve.
Thus a hyperbola is "hemmed in"
between its asymptotes whereas
a parabola has no asymptotes, but
keeps opening up wider and wider
and thus has a shape quite different from
that of a hyperbola (see p. 47).

is positive,*
as, for example,
$(-3)(-3) = 9$,
we get also
points in the third quadrant,
as shown in Fig. 10 (p. 57) and in
the table on p. 58,
thus obtaining the TWO branches of
the hyperbola in question.
And, for this second branch,
the one in the third quadrant (p. 57),
the curve approaches the X-axis
way off to the LEFT,
"meeting" it at an "ideal point"
where $x = -\infty$
and approaches the Y-axis
BELOW,
"meeting" it at an "ideal point"
where $y = -\infty$.
Thus it SEEMS that
the hyperbola has
TWO "ideal points" in common with
the X-axis,
one at $+\infty$, and another at $-\infty$
(and similarly
TWO "ideal points" in common with
the Y-axis),
BUT
remembering what was said
on p. 37,
these two are considered to be
ONE AND THE SAME "ideal point",
and so we say that
the hyperbola has

*If you want to see a PROOF of this
look it up in
"A Survey of Modern Algebra" by
Birkhoff and MacLane.

only ONE "ideal point" in common with
the X-axis
and similarly it has
only ONE "ideal point" in common
with the Y-axis,
and consequently
the hyperbola has
only TWO "ideal points" in ALL,
or TWO points at infinity.

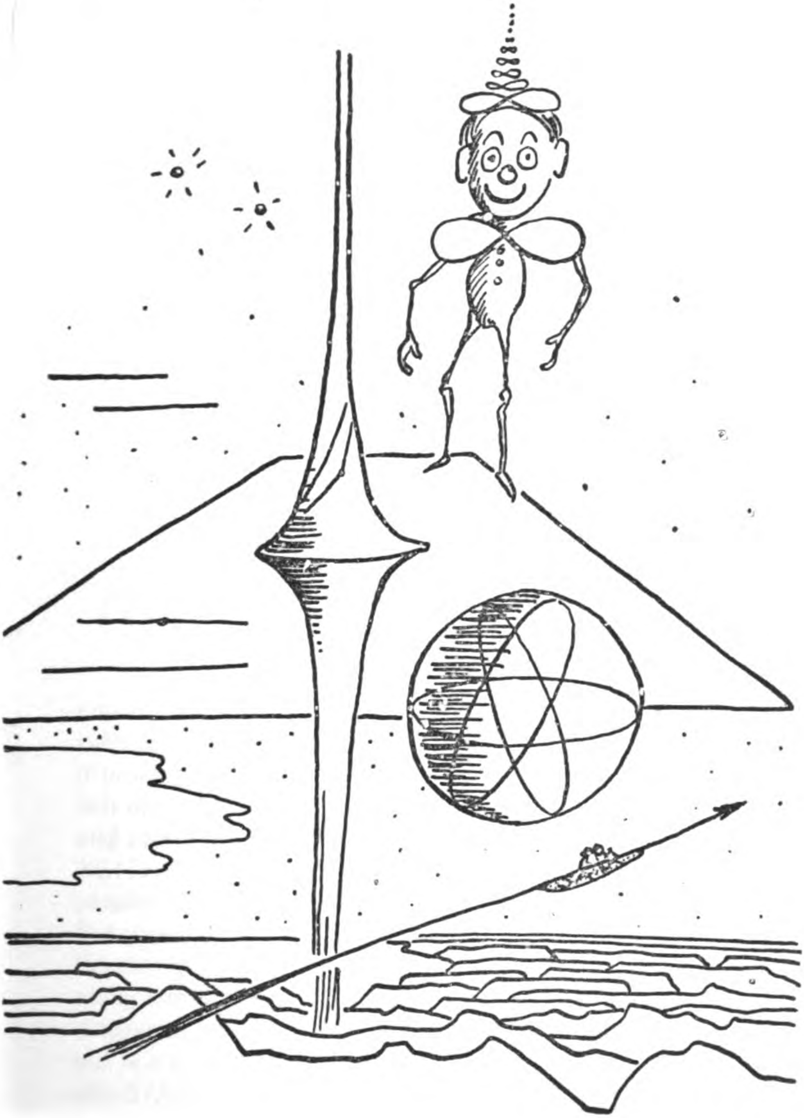
In the next chapter you will see
that this is one reason why
one of the new
NON-EUCLIDEAN geometries
is called
HYPERBOLIC GEOMETRY.

The very idea that
there CAN be
a NON-Euclidean Geometry,
where the Pythagorean Theorem
no longer holds,
where the angle-sum of a triangle
is NOT 180°
(contrary to what you were taught
in school!),
and where many other strange things
happen, —
all this is another example of
the DARING of the HUMAN MIND
as displayed in mathematics,
and, as you will soon see,
it is not just
empty useless bragging about courage,
but the genuine article with
tremendously useful consequences!

5 Non-Euclidean Geometries

On p. 35 reference was made
to one of the basic postulates of
Euclidean Geometry,
namely:
Through a given point which is
not on a given straight line,
ONE and ONLY ONE line can
be drawn which is
PARALLEL to the given line.
Now Euclid regarded
the postulates as
"self-evident truths",
but this particular postulate,
known as the fifth or
"parallel postulate",
did not seem to him to be
so "self-evident", and
he therefore tried to PROVE it from
the remaining postulates.
But he did NOT succeed in doing so,
and therefore
set it down among his postulates.

After Euclid, for many centuries,
other great mathematicians
continued trying to PROVE
this PARALLEL POSTULATE
from the remaining Euclidean postulates,
but also failed to do so.
Although Euclid himself



lived about 300 B.C.,
it was NOT UNTIL about
1826,
that the reason for this failure
became clear!
It finally dawned on
some mathematicians that
postulates are NOT
"self-evident truths" at all,
but are merely
MAN-MADE ASSUMPTIONS!
And hence
may be CHANGED!
Thus
three mathematicians
(quite independently of each other!
and all at about the same time!
around 1826),
namely,
Lobachevsky (a Russian*)
Bolyai (a Hungarian*) and
Gauss (a German*)
undertook investigations
to see what would happen if
Euclid's Parallel Postulate
were changed to read:
"Through a given point which is
not on a given straight line,
TWO different lines could be drawn
BOTH of which are

---

*I mention their nationalities only
because there are people who
think that some nationalities are
"better" than others,
whereas history shows that
various nationalities have
contributed to human knowledge —
and here we have
one such illustration.

PARALLEL to the given line" —
while leaving
all the other Euclidean postulates
UNchanged.
Perhaps this idea seems to you
even now
to be TOO fantastical,
for you may be visualizing
a figure like this:

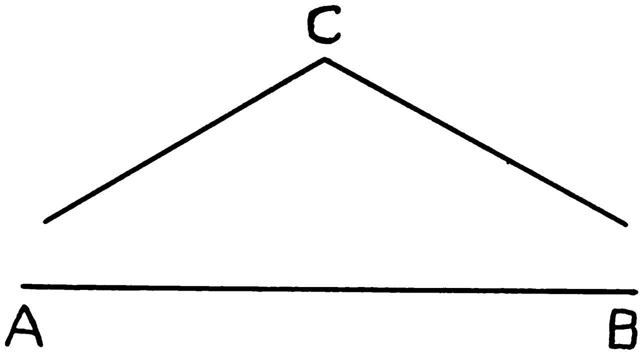


Figure 11

and thinking that
both lines through C,
if prolonged,
will meet AB (prolonged) somewhere,
and therefore
NEITHER of them can possibly be
parallel to AB!
But you should know that
Geometry
really has nothing to do with
diagrams,
but is a subject in which
one STARTS with POSTULATES
(which must not contradict each other)

and from them one derives
THEOREMS (or consequences)
by means of LOGIC.
Does this thought give you no comfort,
and are you still worrying about
Fig. 11?
But just wait and see!

Now, the three mathematicians
mentioned above (p. 64),
starting with that strange
NEW PARALLEL POSTULATE
(according to which:
through a given point not on a
given line
TWO lines can be drawn
BOTH parallel to the given line),
derived from it
(while leaving the other postulates of
Euclid unchanged — see p. 65)
some very strange theorems
(for example,
the angle-sum of a triangle is
LESS than 180° —
instead of being EQUAL to 180° as in
Euclidean Geometry,
which we all learned in school) —
still there was
NOTHING CONTRADICTORY
WITHIN the new system!
And consequently
it was just as good a geometry as
the one to which
we have all been accustomed.
"But", perhaps you will say,
"what IS the angle-sum of a triangle,
is it 180° or not?!"

And the answer is that
the only way you can find out is
by "proof",
and "proof" means
deriving consequences from postulates,
so that
in Euclidean Geometry
you find out by "proof" that
the angle-sum of a triangle IS 180°,
whereas
in the above mentioned
Non-Euclidean Geometry
you find out by
just as good a "proof" that
the angle-sum of a triangle is
LESS than 180°.
Are you bewildered?
Before "unbewildering" you,
as in the dénouement of
a detective story,
I shall first have to tell you
something which will make
this whole business
SEEM still worse,
but do not worry,
for it will all come out fine
very soon,
by the end of this chapter!

About 1850,
along came another mathematician,
Riemann (a German),
who undertook still another investigation,
namely, —
he wanted to know
what would happen if
the Euclidean parallel postulate

were replaced by the following one:
"Through a point
not on a given straight line
NO line could be drawn which is
parallel to the given line" —
again leaving all the
OTHER Euclidean postulates
UNchanged.
And he found still another
Non-Euclidean Geometry —
one in which
the angle-sum of a triangle is
GREATER than 180°,
and many other "queer" theorems!
You see why I said
things would seem worse before
they got better again!
You can at least appreciate why
these Non-Euclidean geometries
seemed strange when they first came out,
so much so that
Gauss, who was an old man at the time,
did not even publish his results,
fearing that they would
not be "well received".
But the other two men,
Lobachevsky and Bolyai,
being young,
were not so cautious;
and one of them, Bolyai,
wrote to his father,
who was himself a mathematician,
that he had created
a new world
out of his own imagination.

Then, a few years later,

in 1868,
an Italian named Beltrami
took some of the "mystery" out of all this
by finding that
the first Non-Euclidean Geometry
(the one in which it was assumed that
there are TWO parallels to a given line
through the same outside point)
actually applied on a surface called
a "pseudo-sphere",*
and that the
Riemannian Non-Euclidean Geometry
applied on an ordinary sphere.*

Thus you see that
whereas on an ordinary
"flat" blackboard or piece of paper,
our time-honored Euclidean Geometry is
as good as ever,
but
on a different surface,
like a sphere or a "pseudo-sphere",
you would naturally expect
a DIFFERENT geometry to apply.

And what is the USE of all this?

Well, at least you will agree that
a flyer would be more interested in
the geometry on

---

*As a matter of fact,
these geometries
do not apply EXACTLY on
these surfaces,
but the "application" is near enough
to serve the purpose here.
But see page 146 in
"Non-Euclidean Geometry" by
Bonola.

the surface of a sphere
(since, no matter how high
he has been able to fly until now,
he still follows the "curvature"
of the earth)
rather than in the geometry on
a flat surface.
And you may be interested to know
that the geometry which applies in
modern physics
(Einstein's General Theory of Relativity'
is Non-Euclidean rather than
Euclidean!*

And thus you see that
in the Non-Euclidean Geometry
in which there are
TWO distinct "straight lines"‡
through the point C,
one to the right
and
the other to the left,
BOTH parallel to AB,
each of these TWO lines will have
an "ideal point" in common with
AB prolonged in both directions,
so that a "straight line", like AB,
in this geometry,
has TWO "ideal points"

---

*See my little book on
The Einstein Theory of Relativity.

‡I put quotation marks here because
on these curved surfaces
there are no straight lines at all
but merely shortest distance paths
called GEODESICS:
thus for example on a sphere
a great circle is the
shortest distance path between
two points.

just like an ordinary HYPERBOLA
(see p. 61)
instead of only ONE "ideal point" as
does a straight line in
Euclidean Geometry (see p. 36).

Hence
the Non-Euclidean Geometry of
Lobachevsky-Bolyai-Gauss
may be called
HYPERBOLIC GEOMETRY.*

And, similarly,
since in the Non-Euclidean Geometry of
Riemann,
through a point C
which is not on a given line AB,
NO lines can be drawn which are
parallel to the given line,
so that a "straight line" here
has NO "ideal points",
hence
this Non-Euclidean Geometry
may be called
ELLIPTIC GEOMETRY,
because
an ELLIPSE, too,
has NO points at infinity,
and thus no "ideal points".

Finally,
since a straight line in
Euclidean Geometry
has ONE and ONLY ONE "ideal point"
(see p. 37),

---

*For another reason for this name
see my "Non-Euclidean Geometry".

just like a PARABOLA,‡
hence
ordinary Euclidean Geometry
may be called
PARABOLIC GEOMETRY.

These terms:

(1) HYPERBOLIC GEOMETRY
(2) ELLIPTIC GEOMETRY
(3) PARABOLIC GEOMETRY

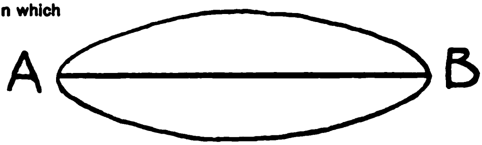
now commonly used to describe

(1) the Lobachevsky-Bolyai-Gauss type
of Non-Euclidean Geometry
(2) the Riemann type
of Non-Euclidean Geometry
(3) ordinary Euclidean Geometry,

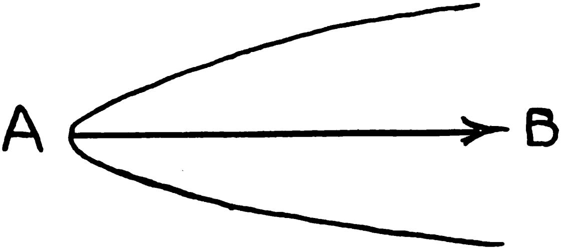
respectively,

---

‡For a parabola may be thought of as
an ellipse in which



the end, B, of the major axis AB,
has moved off to infinity, thus:



were first introduced by
the German mathematician,
Felix Klein,
who not only named
these three geometries
but showed that they were all
"merely different aspects of
a more general kind of geometry
which includes them as special cases."*

And so, to summarize this chapter,
you see that

(a) in geometry, too, as well as in
arithmetic and algebra,
there is a "potential" infinity;
and

(b) the basic postulates of the
three geometries mentioned above
are all alike except only
the parallel postulate, which is
DIFFERENT in each of these
three geometries:
thus,
through a given point, C,
outside a given straight line, AB,
the parallel postulate states that in
(1) hypberbolic geometry
there are TWO lines
BOTH parallel to AB;
(2) elliptic geometry
there are NO lines
parallel to AB;

---

*See "Men of Mathematics" by E. T. Bell
(Fourth Printing p. 379)
and also my little book on
"Non-Euclidean Geometry".

(3) parabolic geometry
there is ONE and ONLY ONE line
parallel to AB.

(c) It is now clear why,
during all those centuries,
mathematicians were
UNABLE TO PROVE
Euclid's parallel postulate —
for it simply DOES NOT FOLLOW
from the OTHER postulates,
but is quite INDEPENDENT of them,
for, as you saw,
it can actually be CHANGED
and STILL be in harmony with
all the other Euclidean postulates,
thus giving rise to
NEW GEOMETRIES which are
just as good as the Euclidean one
but useful for OTHER purposes.

We can see in all this
at least two morals:

(1) Changing only one postulate
(e.g. the parallel postulate) —
and leaving all the others
UNCHANGED,
was enough to yield the
new Non-Euclidean geometries
which have
THEOREMS which are
vastly different from those in
our old Euclidean Geometry
and yet are just as useful.
Thus PROGRESS can be made by
a SLIGHT BASIC CHANGE,

without throwing our
entire heritage
on to an ash-heap!

(2) This progress took
a great many centuries to achieve!
Why, then,
do we get so impatient with
the "United Nations"
and say that
this organization has
accomplished so little in
a few years!
Of course this must not
be interpreted to mean that
we can wait for centuries before
international relations,
and indeed
human relations generally,
are improved!
For:
(1) in the first place,
since modern warfare
has become so DESTRUCTIVE,
it has become completely
out of the question to
solve ANY problem
by this means,
for there will be NO VICTOR.
It has become indeed a
"race between education and
catastrophe" as
H. G. Wells remarked;
and
(2) in the second place
we must realize that
the progress made by SAM,

as, for example, in mathematics,
was not done merely by
waiting and hoping that
something will just "happen"
of its own accord,
but rather, that
SAM has always, and
MUST NOW MORE THAN EVER
GIVE THE BEST THAT IS IN HIM
of "Realism" ("S"),
"Intuition" ("A"), and
"Sweet Reasonableness" ("M")
in order to make progress in
human relations, in
international relations, in
the conquering of
disease and poverty, and
all the other evils that
we have "fallen heir to".

6 "Actual" Infinity

And so you have seen
several illustrations of
"potential" infinity,
as being something which is
APPROACHED but NEVER REACHED.

Consider now
the positive integers,
i.e. the whole numbers,
like 1, 2, 3, etc.
Instead of writing "etc." so soon,
you can of course
continue the sequence
AS FAR AS YOU PLEASE
and then follow it by dots,
like this: . . . .
to indicate that
this sequence is approaching ∞.
BUT,
if you consider this sequence
AS A WHOLE,
you have an illustration of an
"Actual" infinity.
Perhaps you think that
you cannot do anything with
such a class or set of objects,
you cannot even NAME them all,
for, no matter how far you go,
there are still
more and more and more of them
coming up.
You will soon see however

that a great mathematician,
Georg Cantor,*
was able
not only to DO something with
this "actual" infinity, but
to make up a whole SYSTEM of
"actual" infinities,
creating what is known as
Cantor's Mengenlehre
(i.e. Cantor's Theory of Sets),
which I shall try to tell you about.

But first
let me put a simple question to you:
Suppose you went to a stadium
the capacity of which is known to be
50,000;
and suppose you looked around
and saw that
every seat was taken and
that no one was standing
(you had your own seat of course).
Now if you were asked
"Can you tell me
how many people there are
in this stadium
WITHOUT COUNTING THEM?"
You would surely answer,
without any hesitation,
"Fifty thousand",
would you not?
But how do you know?
You did not count them!
To which you would

---

*See "Contributions to the Founding of the Theory of Transfinite Numbers" which contains, in English translation, Cantor's two very important papers (1895 and 1897), with an Introduction by Philip E. B. Jourdain.

undoubtedly reply that
since the capacity of the stadium
is 50,000,
and every seat is taken,
and no one is left standing,
there must be 50,000 people,
one for each seat.

Thus, by "pairing off"
each seat with each person
you are able to know that
there are just as many people
as there are seats.
And similarly
if you have any set of objects, A,
consisting of the individual members.
a, b, c, etc.
and another set, A′,
consisting of
a′, b′, c′, etc.;
and if
for every member of A
there corresponds
one and only one
member of A′,
and conversely:
for every member of A′,
there corresponds
one and only one
member of A, —
then you will agree that
A and A′ are
"equivalent",
that is,
they each have
the same number of members.
This pairing off process

is called
one-to-one correspondence,
often written 1-1 correspondence.

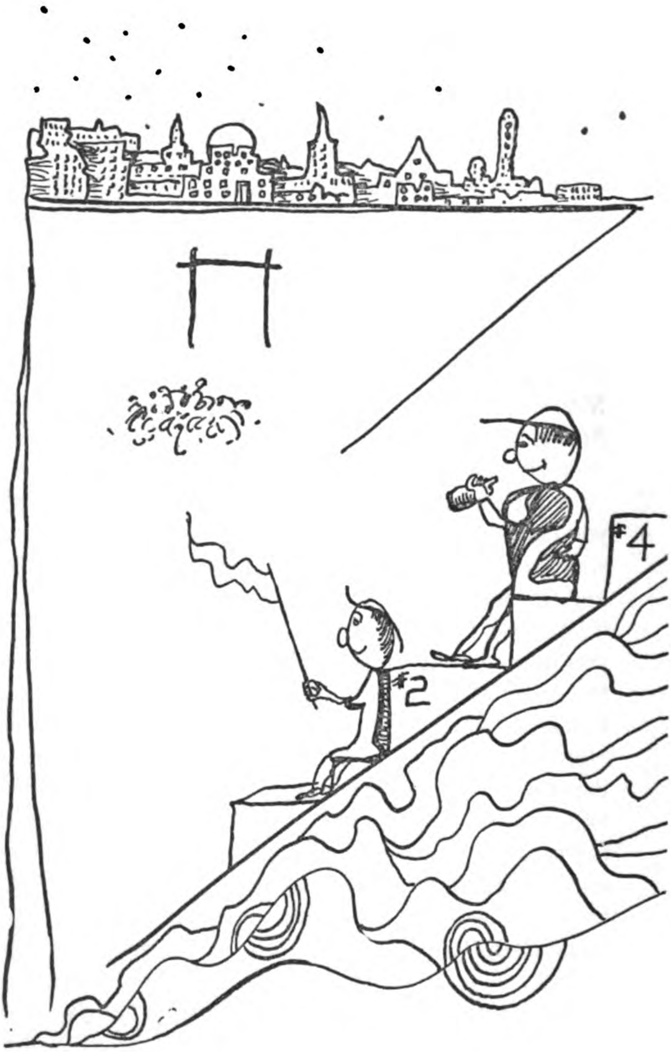
Now let us go back to
the "actually" infinite set of
positive integers.
You will soon see that
it has a remarkable property
that no FINITE set ever has!
For
you will agree that
all the EVEN integers (2, 4, 6, etc.)
are contained in this set,
thus being a PART of
the ENTIRE set of positive integers
(both even and odd).
And yet
let us now compare
the following two sets:

(1) the set of ALL integers:
1, 2, 3, 4, 5, ...
(2) the set of EVEN integers only:
2, 4, 6, 8, 10, ...

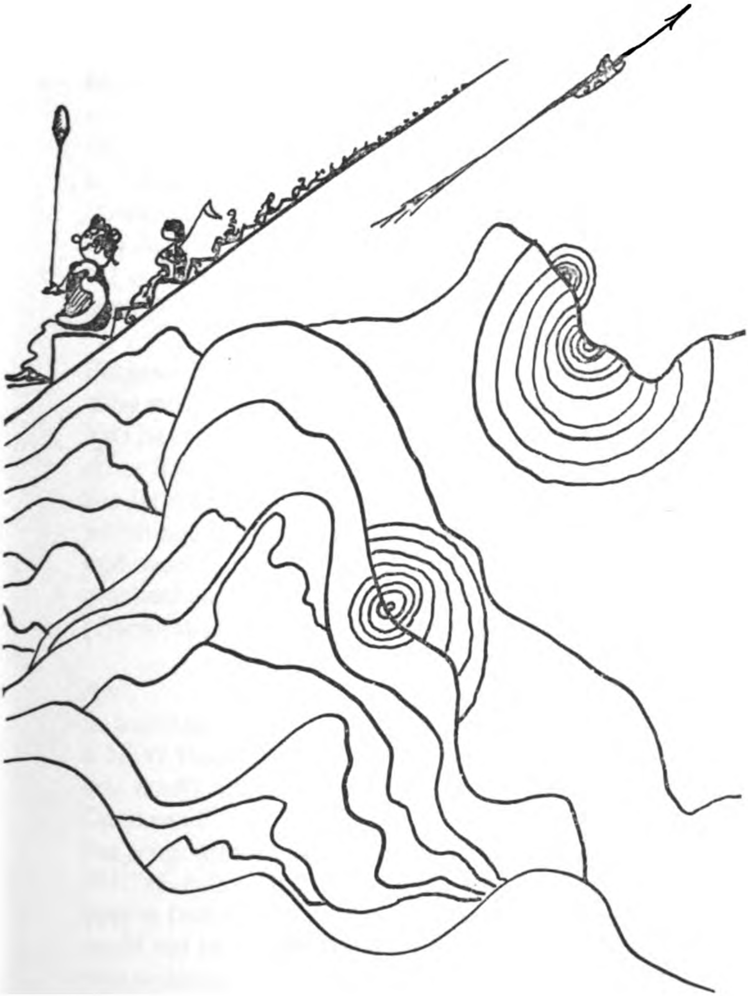
Let us think of set (1) as being
composed of people,
Mr. 1, Mr. 2, Mrs. 3, Miss 4, ...
and set (2) as being
seat-numbers,
seat #2, seat #4, etc., etc.
And now let us assign seats
to the people
in such a way that
Mr. 1 is to occupy the first seat,
that is, seat #2;

Mr. 2 shall have the next seat,
that is, seat #4;
Mrs. 3 has seat #6,
Miss 4 has seat #8,
etc., etc.
Thus you see that
each person is assigned to a seat
whose number is
double the number which represents
his own name,
so that
Monsieur 9 will occupy seat #18,
and seat #22 will be occupied by
Madame 11.
And similarly
as soon as you have
"a person's number"
you can immediately tell
what seat is reserved for him or her,
and, conversely,
if you look at any seat number
you immediately know
the name of the person to whom
this seat belongs.
Hence
there is a 1-1 correspondence
between the members of
the sets (1) and (2) on p. 80,
and consequently
these two sets are
"equivalent",
that is,
they each have
the same number of members,
as you agreed on p. 79.

Perhaps you are thinking,



#4
#2



"How can this be?!
Set (2) is a PART of set (1),
so how can they both be
the SAME 'size'?
Is it not true that
'The whole is GREATER than
ANY of its PARTS',
as I learned in school,
and as I see in
the practical world
every day?"

To which the answer is:
"What you learned in school
still applies,
as much as ever,
BUT ONLY to FINITE sets —
thus you will not be able
to establish a
1-1 correspondence between
a finite set of, say, 7 objects
and another finite set of, say,
4 objects;
when you pair them off,
there will always remain
3 of the 7 left out, thus;

(1) 1, 2, 3, 4, 5, 6, 7
↕ ↕ ↕ ↕
(2) 1, 3, 5, 7

But you see that
if the two sets are each
"actually" infinite sets,
like the sets (1) and (2) on p. 80,
EVERY member of the first set
can have a mate of

the second set,
and conversely;
so that here
the sets CAN be "equivalent"
even though one is only
a PART of the other,
as shown on p. 81.
Indeed one of the basic ideas
in dealing with these
"transfinite numbers" is that
a "transfinite number"
(UNLIKE a finite number)
can ALWAYS be put into
1-1 correspondence with
some PART of itself.

Fantastic?
Why not?
You have already had
other instances of
the fantastic in mathematics,
which has led to progress
and even to
practical applications.
(Remember the Non-Euclidean Geometries?)

And,
in building up
a NEW theory of transfinites
you would not expect that
Cantor use
the postulates appropriate for
FINITE sets ONLY,
just as Non-Euclidean Geometries
could not be built up if
you retained
ALL the Euclidean postulates

(including the parallel postulate;
see pp. 66 ff.)!

Thus each "system" of thought
must have its own postulates,
appropriate to it.
BUT of course it is ESSENTIAL that
the postulates WITHIN a system
should NOT CONTRADICT each other.
Naturally each system
would say to the other:
"You look queer to me,
but if we each stay in
our own bailiwick
and apply ourselves to
our own problems,
we can both co-exist
and make the world richer
by both your presence and mine."
Surely the various geometries
can say this to each other,
and similarly the theories of
FINITE sets and
"actually" INFINITE sets
may speak to each other
in the same way,
NOT EXPECTING to be IDENTICAL
with each other,
and not even wishing to be so,
since the loss of either one
would be a loss to the world.

And now let us look at
some other
"actually" infinite sets.
You will find them to be
MOST INTERESTING.

7 "Actual" Infinity (cont.)

Consider next
the set of rational numbers,
another example of an
"actual" infinity.
But first you must know that
the word "rational" when applied to
numbers
has nothing whatever to do with
the word "rational" when applied to
people!
Words are "tricky" and
we must beware of them!
A PERSON is rational when
he uses his "M" properly
(Remember the M in SAM?);
but a number is rational when
it can be expressed as
a RATIO (hence the term RATIOnal)
of two INTEGERS,
thus 7/8 is a rational number
because
7 and 8 are INTEGERS and
the division line between them
expresses the idea that
7/8 is the RATIO of 7 and 8.
Similarly
2 and 1/4 is a rational number
because
it may be written 9/4
which is
the RATIO of the INTEGERS
9 and 4.

You can see that
an integer, like 11,
is itself a rational number
since it can be written 11/1.
Thus the set of rational numbers.
INCLUDES the integers.

Now you know that
the sum of
any two rational numbers
is also a rational number:
thus

$$1/3 + 2/5 = 5/15 + 6/15 = 11/15$$

and, in general,

$$a/b + c/d = ad/bd + bc/bd = (ad + bc)/bd$$

where a, b, c, and d are integers
and hence
$ad + bc$ and bd are also integers,
so that the answer is
again a rational number.
This simple fact leads to
a most STARTLING result!
For you can see that
if you take any two rational numbers,
say, 1/2 and 2/3,
it is possible
to find another rational number
BETWEEN them
(that is,
larger than 1/2 and smaller than 2/3)
in various ways,
as, for example, by
finding the "average" of the two,
thus:

$$1/2 + 2/3 = 3/6 + 4/6 = 7/6$$

and dividing this result by 2
to get the "average",
we obtain

$$7/6 \div 2 = (7/6)(1/2) = 7/12.$$

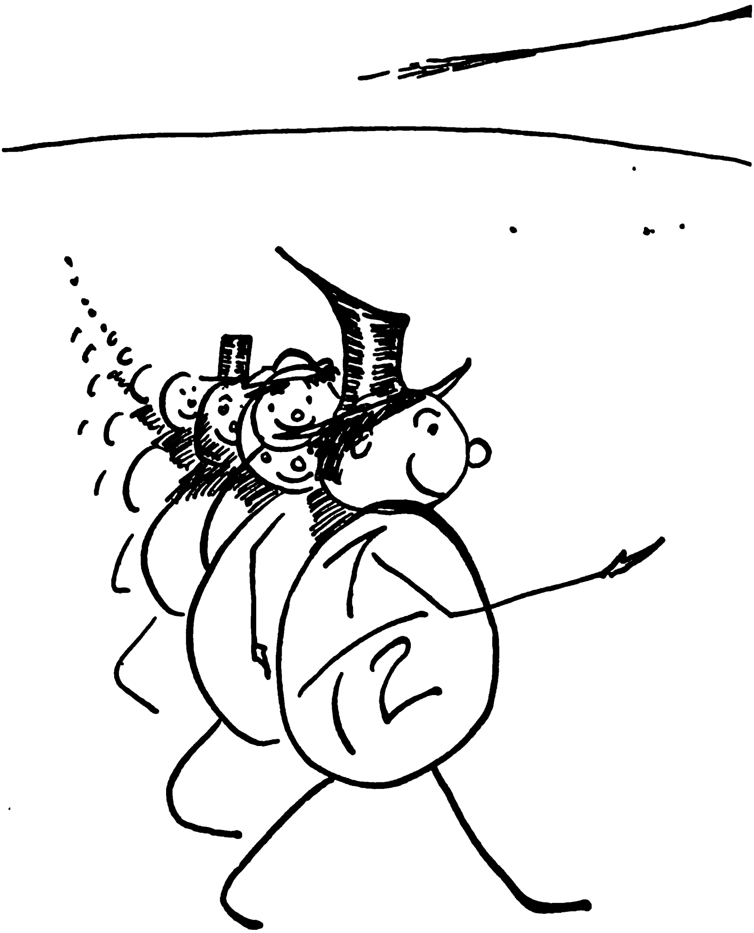
And 7/12 is
larger than 1/2 (which is only 6/12)
and smaller than 2/3 (which is 8/12).
Now what is so startling about this?
Well, you see,
you can now repeat the process
and find another rational number
between 1/2 and 7/12,
and then another
between 1/2 and this result,
and so on and so on,
thus obtaining
an INFINITY of rational numbers
between the 1/2 and 2/3
with which you started.
And so,
BETWEEN ANY TWO RATIONAL NUMBERS
you can "pack in"
an INFINITY of other rational numbers,
so you see that
the entire set of
positive rational numbers
is of a very different nature from
the infinite set of INTEGERS.
For in this latter set
although there is
an "actual" infinity of them.
as was said on p. 77,
still you cannot "pack in"
any integers BETWEEN
two successive integers,
say between the integers 2 and 3.

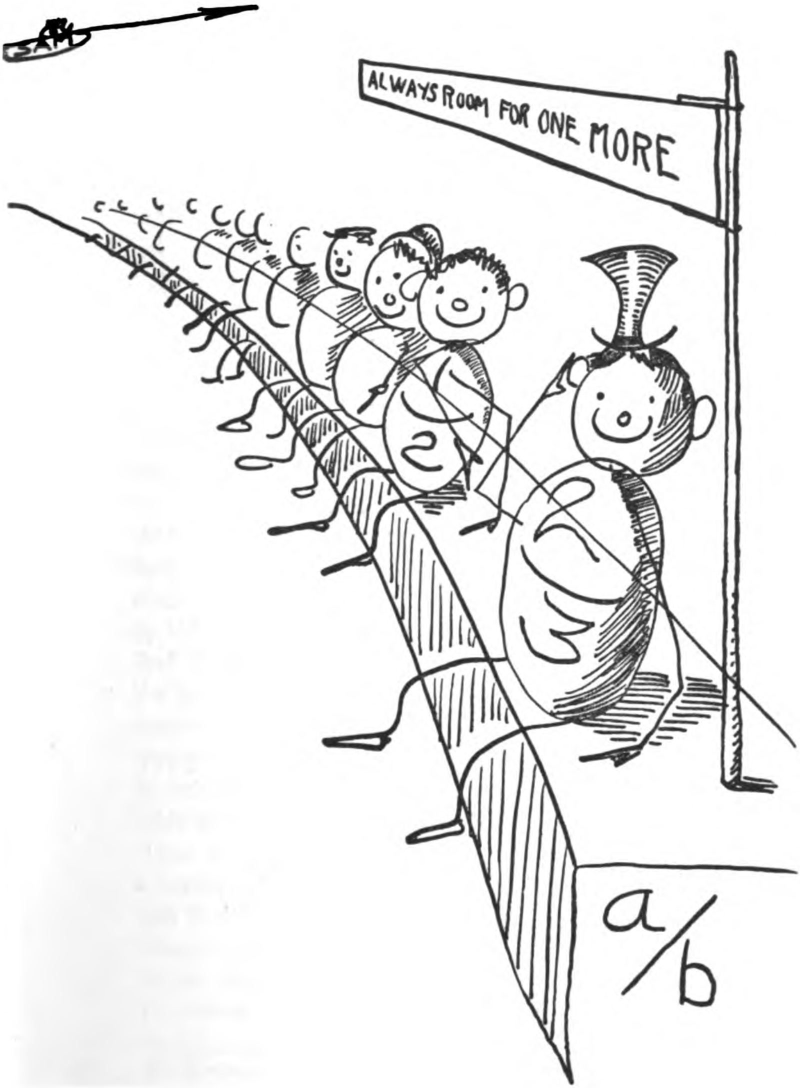
Thus
the actual infinity of integers is
a "discrete" set,
that is,
if you take any integer, like 7,
it has an IMMEDIATE successor, —
in this case, 8.
Such an infinite set is called
a "countable" or "denumerable"
infinite set,
since you can at least
start counting them,
going from one to the NEXT, etc.,
even though you can never finish
the process,
since it goes on and on.
But an infinite set like
the rational numbers,
in which any given rational number,
like 1/2,
does NOT have an IMMEDIATE successor
(since any rational number you may pick
that is larger than 1/2,
no matter how near to 1/2 it is, —
you can always "pack in"
an infinity of other rational numbers
between it and 1/2)!
Such an infinite set
is called
a DENSE set.

Now if you were asked
"Do you think that
the infinite set of rational numbers
is 'equivalent' to
the infinite set of integers,
just as the set of ALL integers

was shown on p. 81 to be
'equivalent' to
the set of EVEN integers ONLY?" —
what would be your answer?

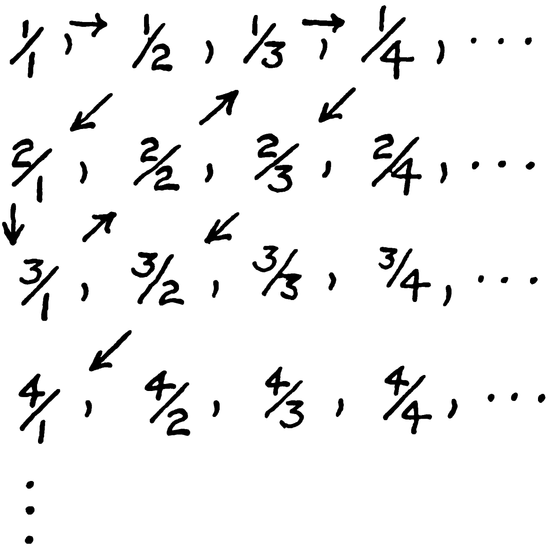
Perhaps you would be inclined
to reason like this:
Since the infinity of integers is
a DISCRETE set,
as was said on p. 90,
whereas the infinity of
rational numbers is
a DENSE set (p. 90),
how is it possible to establish a
1-1 correspondence between them,
as was done for
the two discrete sets on p. 80?
That "equivalence", on p. 81,
was surprising enough,
since it showed that,
for infinite sets,
a WHOLE set can be
equivalent to a PART of itself!
But now
a still greater surprise
awaits you,
for you will presently see that
the set of all rational numbers,
although it is a DENSE set,
can also be put into
1-1 correspondence with
the set of all integers,
although this latter set is
a DISCRETE set!
In order to do this, however,
it will be necessary to
arrange the rational numbers




SAM
ALWAYS ROOM FOR ONE MORE
a/b

in an order DIFFERENT from
the USUAL order of
increasing magnitude.
And this new arrangement
will be such that
the set of all rational numbers
will be seen to be capable of
being put into an order which
will make it a
DISCRETE set,
in which
any given rational number
DOES have
an IMMEDIATE SUCCESSOR.

This can be done in various ways,
of which the following is one:



Note that in the first row are
all the rational numbers whose

numerator is 1, and
whose denominators are
1, 2, 3, 4, etc.;
in the second row are
all the rational numbers whose
numerator is 2, and
whose denominators are
1, 2, 3, 4, and so on up;
similarly for the third row and
for all subsequent rows.
Now first of all
it is obvious that
all the positive rational numbers are
included in this set,
and yet
it is now quite easy to show that
this is a DISCRETE set,
for, by following the arrows,
we can go from one rational number
to the NEXT,
since now each one has
an IMMEDIATE successor.
And thus we see that
the set of all
positive rational numbers,
though DENSE when arranged
in order of increasing magnitude,
CAN be RE-ARRANGED so that
it can be shown to be
a "countable" or "denumerable" set,
just like the infinite set of
integers, with which it can now
be put into
1-1 correspondence!*

*Although in this arrangement,
the fifth number, 2/2,
has the same value as
the first, 1/1,

But this is only
the BEGINNING of
the amazing subject of
the "actual" infinities
as developed by
the great Georg Cantor.
You will see later how useful
and in fact INDISPENSABLE
it has become for mathematics! —
both "pure" and "applied".
And you will also see how
Cantor's Theory
was attacked! —
not only on the ground of
being TOO fantastic,
but it was even shown
to lead to
INCONSISTENCIES,
and this latter objection is indeed
INTOLERABLE —
for inconsistency is
the one unforgivable sin in
mathematics!

You will not be surprised that
Cantor's Theory of "actual" infinities

---

and we are thus counting
the same number more than once,
so that the correspondence
does not seem to be 1-1,
still we need not worry about this,
for we need only
ELIMINATE all duplicates
as we come to them,
and thus NOT count them more
than once.
Thus, by eliminating 2/2,
the next one, 1/3,
becomes the fifth number,
etc., etc.

was NOT REJECTED because of its
"fantasticalness" —
for, as you already know,
mathematics and FANTASY make
very good companions indeed.
But you will undoubtedly want to know
what has been done about
the inconsistencies or "paradoxes",
for these really
CANNOT be tolerated,
since, as you know full well,
no game can be played with
contradictory rules,
and similarly
no reasoning can be done with
contradictory statements.
When any such are found,
SOMETHING MUST BE DONE ABOUT IT!
Some possible remedies that
are sometimes used are:

(1) the postulate set may have to be
CHANGED
as Einstein did in
constructing his
Theory of Relativity,
when a contradiction arose in
Physics.*

(2) A postulate may have to be
ADDED,
as was done in
Euclidean Geometry
when a contradiction arose.‡

---

*See my "The Einstein Theory of Relativity".

‡See "The Education of T. C. Mits"
by L. R. Lieber, page 135.

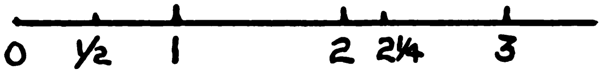
(3) It was suggested by
some mathematicians
to discard entirely
the whole Cantor Theory
as being the product of
an insane mind!
But as it is realized
more and more
how ESSENTIAL this theory is
for all of mathematics
as well as for its
practical applications,
this "remedy" is seen to be
a destroying poison and
not a cure at all.
You will see,
before you have finished
reading this little book,
what was done about it!

And what an amazing story it is!
For the powerful METHODS which
mathematicians and logicians
use
to eliminate or minimize
the difficulties which
they encounter,
are an inspiration for
ALL OF US
in helping us to see
what the human mind is
CAPABLE of doing
when it runs into trouble,
to see our greatest
STRENGTH
as well as our
WEAKNESS,

so that we may know our
LIMITATIONS,
beyond which we cannot go,
and yet to see
how VERY far we CAN go,
if we gain insight into the
POWERFUL
STREAMLINED
MODERN
methods of
SAM!

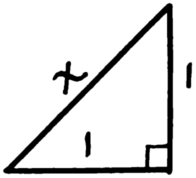
8 A Still Greater "Actual" Infinity!

Suppose you now imagine
the positive rational numbers
placed on a line
in order of magnitude,
like this:



together with all those
that are "packed in" (see p. 89).
You might think that
since you can "pack in"
as many as you please
between any two of them,
that therefore
this dense set of numbers
covers the line completely,
and that any point on the line
would therefore correspond to
a definite rational number.
BUT THIS IS NOT SO!
For I shall soon show you
a point on the line
which has NO RATIONAL NUMBER
to designate it
and therefore there is
a "gap" at this point!
And indeed it turns out that
the line has MORE GAPS

than it has points which
DO have RATIONAL NUMBER designations!
The line is like a sieve! —
full of holes!
Let me show you
one such gap:
consider the right triangle



each of whose legs is ONE unit long;
then the length of the hypotenuse, x,
can be found by the well-known
Pythagorean Theorem,
thus:

$$x^2 = 1^2 + 1^2$$

therefore

$$x^2 = 1 + 1$$

or

$$x^2 = 2$$

and hence

$$x = \sqrt{2}.$$

Now, if you go back to
the line on p. 100,
on which
the distance from 0 to 1
is the same as the unit length
of each leg of the triangle above,
and now lay off, from 0,
a distance equal to
the length of the hypotenuse of
this triangle,

you will obtain a point
whose distance from 0
is equal to $\sqrt{2}$,
will you not?

And now I shall prove to you
that $\sqrt{2}$ is
NOT a RATIONAL NUMBER,
that is,
it CANNOT be expressed by

$$a/b$$

where a and b are INTEGERS!
For,
suppose that it COULD be
so expressed, thus:

$$\sqrt{2} = a/b \tag{1}$$

and suppose that
a/b is in its LOWEST terms,
that is,
a and b have
NO FACTOR IN COMMON,
all such common factors having been
already cancelled out
in reducing the fraction to its
lowest terms.
Now,
if equation (1) were true,
we could then obtain,
by squaring both sides,

$$2 = a^2/b^2;$$

and, multiplying both sides by b^2,
we would get

$$2b^2 = a^2.$$

Here, the left-hand side, $2b^2$,
is obviously an EVEN number,
since it has the factor 2 in it.
Consequently,
the right-hand side, a^2,
being equal to $2b^2$,
must also be EVEN.
But if a^2 is EVEN
then a itself must also be EVEN.*
Hence EACH a in a^2 is EVEN,
and therefore
a^2 is not only even but
is exactly divisible by 4
(since there is a 2 in EACH a).
Hence $2b^2$ must also be
divisible by 4,
and since we SEE only one 2 in it,
the other 2 (to make up the 4)
must be a factor of the b^2,
so that b^2 must be EVEN,
and hence also b must be EVEN.
Thus you see that
we have been obliged to admit that
BOTH a and b are EVEN
and therefore have the factor 2
IN COMMON —
which CONTRADICTS the statement
on p. 102
that
a and b have
NO FACTOR IN COMMON!
And consequently
the assumption that

$$\sqrt{2} = a/b$$

---

*For, an odd number, $2n + 1$,
when squared,
gives $(2n + 1)^2 = 4n^2 + 4n + 1$
which is also odd.

where a and b are
two INTEGERS having
NO common factor,
is a FALSE assumption,
which is to say that
$\sqrt{2}$ is NOT a RATIONAL NUMBER,
and yet there is a point on the line (p. 100)
which has the designation, $\sqrt{2}$
as explained on p. 101.
Such a number is called
an IRRATIONAL NUMBER,
meaning that it is
NOT a RATIONAL NUMBER.
And similarly for all
the other "gaps" on the line.
Now if we fill up all the "gaps"
and call such numbers the
irrational numbers,
then the TOTALITY of
rational AND irrational numbers
together,
are called REAL numbers.
Thus the real numbers constitute
a "continuous" set of points
or a "continuum".
Now you will see later that
mathematicians have learned
through the "hard knocks" of
EXPERIENCE
that this kind of
intuitive geometric reasoning
(using the "line" on p. 100,
for instance)
has SOMETIMES
played them false,*

---

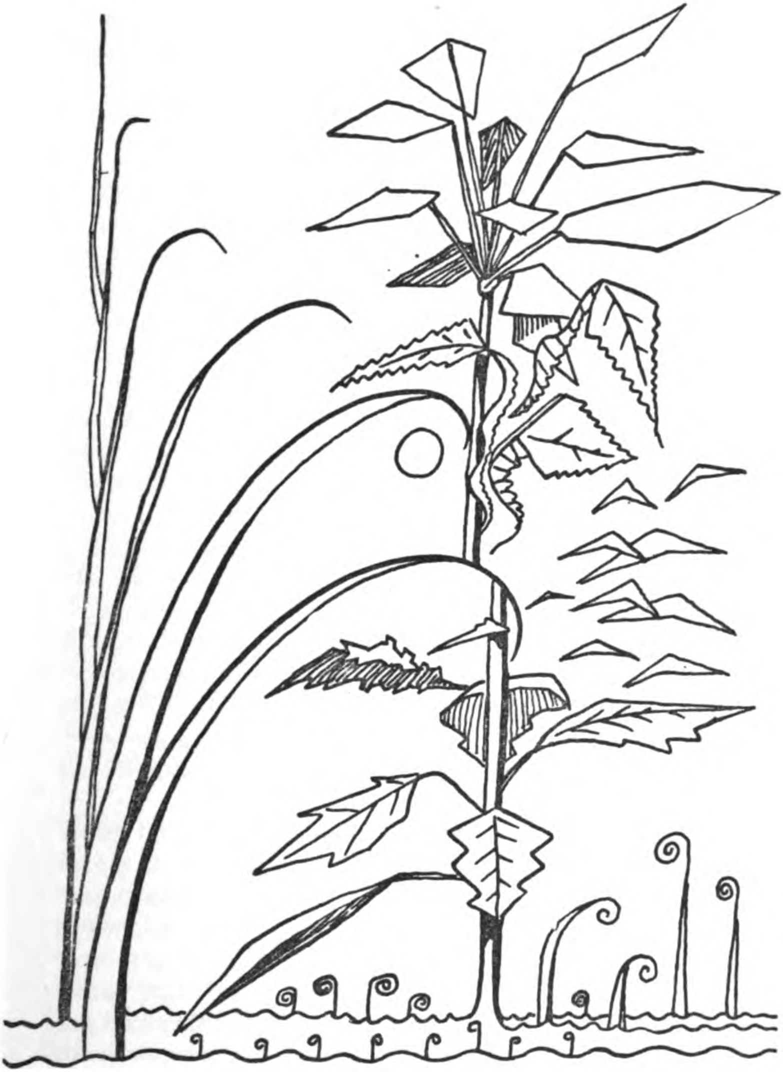
*See the second footnote on p. 240.

and therefore,
during the 19th century,
the definition of
"real" numbers
was made more rigorous,*
being made to depend upon
rational numbers, which in turn,
as you know,
depend on INTEGERS only,
and NOT on a GEOMETRIC concept like
a line.
But for our purpose here
the idea of associating
the real numbers with
points on a line
will be very helpful
and not misleading.
It is obvious
first of all
that there is
an "actual" infinity of real numbers.
And it is natural to ask now:
Is this again
a "countable" or "denumerable"
infinity
as the DENSE set of
RATIONAL NUMBERS
turned out to be (see p. 95)?
That is,
can the set of REAL numbers
be so arranged that
they may be put into
1-1 correspondence with

---

*See p. 240, also pp. 196 ff., and
for a rigorous treatment of
real numbers,
see "The Theory of Functions of
a Real Variable" by E. W. Hobson.

the integers,
or not?
You will see in the next chapter
how Georg Cantor
PROVED that
the set of real numbers
is NOT a "countable" set,
but that here we have
an "actual" infinity which is
GREATER than
a "countable" or "denumerable"
infinity!
And you will see that
in Cantor's Theory of Sets
there is
a whole HIERARCHY of
"transfinite numbers",
getting larger and larger
without end,
and yet
being subject to
definite postulates
permitting you
to operate with these
"numbers"
in a legitimate and
useful manner!
David Hilbert,
himself one of the
greatest mathematicians of
recent times (1862-1943),
considers this achievement of Cantor's
to be:
"die bewundernswerteste Blüte
mathematischen Geistes und
überhaupt eine der höchsten
Leistungen rein verstandesmässiger



menschlicher Tätigkeit"*
(that is,
the most wonderful flowering of
the spirit of mathematics,
and indeed one of the greatest
achievements of human reason).

And now let us look
a little more closely
at the set of REAL numbers,
represented by the points on a
"continuous line".

---

*David Hilbert: "Über das Unendliche",
Math. Annalen, 95 (1926), p. 167.

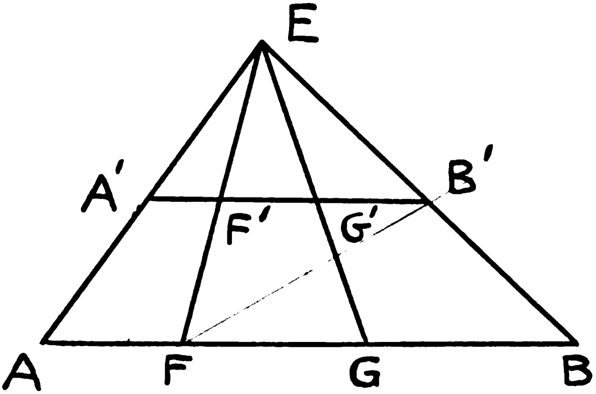
9 The Continuum of Real Numbers

First of all
consider a line-segment which has
all the rational and irrational points
(that is, all the "real" points)
which are included between
its end-points, inclusive,
and therefore has no "gaps";
such a line-segment is an example of
what is called a
"linear continuum".

Let us now see that,
like all "actual" infinities,
it has the property that
ALL of its points may be put into
1-1 correspondence with the points of
only a PART of itself (p. 85).
This may be easily done in
the following way:

Given the line-segment AB (p. 110);
take a shorter segment A′B′,
then draw lines AA′ and BB′,
prolonging them until they meet, at E.
Now, any point, F, on AB,
has a "mate" on A′B′ which
can be located by drawing FE;
then the point where FE crosses A′B′,
namely F′, can be
considered to correspond to F.

And similarly,
any point, G′, on A′B′,
has a "mate" on AB
which can be located by
drawing EG′ and prolonging it until
it meets AB, say at G.



Then G corresponds to G′.
Thus EVERY POINT on AB
has a "mate" on A′B′,
and conversely,
EVERY point on A′B′
has a "mate" on AB,
so that
the two sets of points,
AB and A′B′,
are easily put into
1-1 correspondence —
hence these two sets are
"equivalent".
This "equivalence" may also be shown
analytically (see Appendix C1, p. 356).

And of course
in the same manner
we can also establish a
1-1 correspondence between
A′B′ and AF (p. 110)
which are then also equivalent.
Hence we now have a
1-1 correspondence between
AB and AF (a PART of AB).

Thus any linear continuum,
no matter how long or how short it is,
has the SAME "number" of points on it,
and this "transfinite number" is
designated by c.

We shall soon see (p. 115) that
c is GREATER than
a "denumerable" infinity which
Cantor designated by
the first letter of
the Hebrew alphabet, $\aleph$, with
the subscript 0,
thus, $\aleph_0$,
to distinguish it from
other larger "transfinites" which
we shall meet later and which
will be designated by

$\aleph_1$, $\aleph_2$, etc., etc.

But first let us see how
Cantor showed that
c is NOT denumerable.*
Since, as was shown above,
the LENGTH of the line-segment

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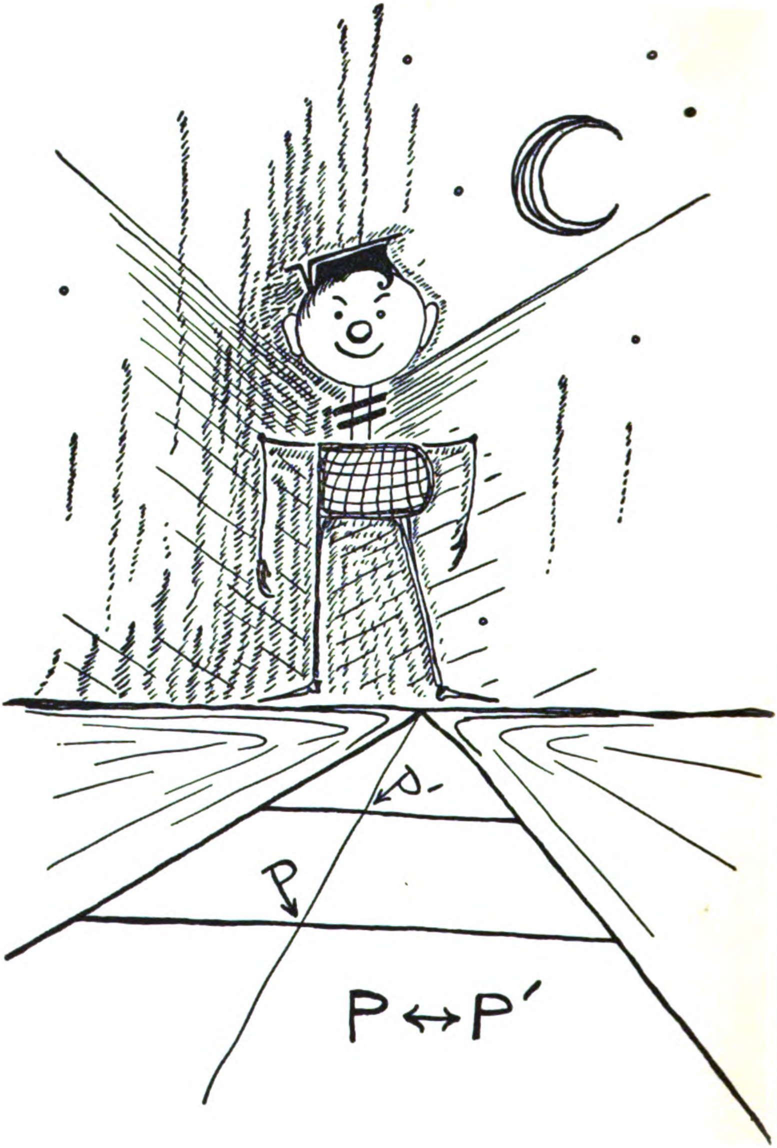
*Jahresbericht der deutschen
math. Vereinig. vol. 1 (1892), p. 75.


P′
P
P ↔ P′

has nothing to do with the case,
let us concentrate our attention on
the line-segment from 0 to 1,
which will therefore contain
the real numbers in this interval,
and each such number may be
represented by an infinite decimal,[*]
thus:

$$0.\varepsilon_1\varepsilon_2 \ldots\ldots$$[‡]

where each ε represents one of the
digits from 0 to 9 inclusive.

Now suppose that the set of
all real numbers in this interval,
from 0 to 1,
WERE "countable";
we could then list them
and number them:
the first, the second, etc., etc.,
thus:

(1) $0.a_{11}a_{12} \ldots\ldots$[‡]
(2) $0.a_{21}a_{22} \ldots\ldots$
(3) . . . .
. . . .
. . . .

---

[*] Thus even the finite decimal, 0.1,
may be considered as the
INFINITE decimal 0.10000 . . . . . ,
the zeros going on to infinity.
See E. W. Hobson: "The Theory of Functions
of a Real Variable" (Cambridge Univ.
Press), vol. 1, pp. 47ff.

[‡] Note the interesting use of subscripts,
single and double:
single subscripts suffice when,
for example,
there is a single sequence,
say, a single row of ε's,
in which you wish to distinguish
between the various terms;
whereas

this list being supposed to contain
ALL the real numbers in this interval.
BUT,
Cantor argued,
after considering this entire set as
being here enumerated,
you can now easily make up
ANOTHER infinite decimal which is
surely NOT in the list:
for you can make up a number which
DIFFERS from the first one in the list
in the FIRST decimal place
(namely, our new number
does NOT have a_{11} in its
first decimal place),
and DIFFERS from
the second number in the list
in the SECOND decimal place
(that is, our new number
does NOT have a_{22} in its
second decimal place),
and so on and so on,
being thus made up to
DIFFER from EVERY number in the list
by going down the diagonal and
changing the digit in
the decimal place you arrive at.

And so,
after thinking it IS possible to

---

when you have many sequences,
then the double subscript
serves to indicate
both the row and the column
in which a particular term occurs,
thus a_{72} above indicates
the digit in the second decimal place of
the seventh number in the list.

ENUMERATE
ALL the real numbers in this interval,
from 0 to 1,
we find that
you can make up NEW ones
which have NOT been included in
the list!

Thus it is IMPOSSIBLE TO IMAGINE
a COMPLETE ENUMERABLE* set of
real numbers in this interval,
or in any other interval.
Hence c is
a "transfinite number" which is
GREATER than $\aleph_0$! (see p. 111)

And we are thus off to a start in
the study of "actual" infinities of
DIFFERENT magnitudes or "powers"!

So far then you have seen that

(1) the "power" of the set of
positive integers
(or "natural numbers",
as they are called)
is $\aleph_0$.

(2) the "power" of the set of
positive rational numbers
is also $\aleph_0$.

(3) the "power" of the set of
ALL real numbers
is c.
Although this was discussed above,

---

*i.e. "denumerable" or "countable."

in the "diagonal" proof
(pp. 114 and 115),
only for the real numbers in
the interval from 0 to 1,
yet it is easy to show that

(a) this set of real numbers in
the interval from 0 to 1
and
(b) the set of ALL real numbers (from $-\infty$ to $+\infty$)
are in 1-1 correspondence* —
and hence if the "power" of (a)
is c,
then this is true also for (b).

Now suppose we should
add to the set of "natural numbers"
also zero and the negative integers,
what would be
the "power" of this new set?
Well,
you can easily see that
by arranging this new set
in the following manner:

0, 1, −1, 2, −2, ...

in which
each positive integer is followed by
its negative,
we again have
a "denumerable" set
of "power" $\aleph_0$.
Similarly the "power" of
ALL rational numbers
(positive and negative and zero)
is still $\aleph_0$.

---

*See Appendix C2, p. 358.

And the "power" of ALL real numbers
(positive and negative and zero)
is still c.*

Now,
before going into a discussion of
OTHER transfinites,
let us take a brief look at
some operations with
those transfinites which have
already been discussed.

It is easy to see that
if you add any finite number,
say 1,
to a "denumerable" set,
the result is still a
"denumerable" set,
since you can start counting
the finite number
and then continue your counting
into the "denumerable" set, thus:

$$\begin{tikzcd}
1 \arrow[d, leftrightarrow] & + (1, \arrow[d, leftrightarrow] & 2, \arrow[d, leftrightarrow] & 3, \arrow[d, leftrightarrow] & 4, \arrow[d, leftrightarrow] & \cdots) \\
1, & 2, & 3, & 4, & 5, & \cdots
\end{tikzcd}$$

or $1 + \aleph_0 = \aleph_0$
and similarly for any finite number, k,

$$k + \aleph_0 = \aleph_0$$

since the k objects
can be numbered from 1 to k

---

*See Appendix C2 p. 358.

and then the count of the $\aleph_0$
can start from k + 1, then k + 2, etc.,
but the totality is still "countable"
and therefore equal to $\aleph_0$.
Indeed,
even $\aleph_0 + \aleph_0 = \aleph_0$, (1)
as you can easily see if
you consider the first $\aleph_0$ to be
the set of positive EVEN integers,
and the second $\aleph_0$ to be
the set of positive ODD integers,
so that
the sum of BOTH
gives you the totality of
ALL positive integers
which is still a "countable" set
whose "power" is still $\aleph_0$.
And equation (1) may be written

$$2\aleph_0 = \aleph_0.$$

And it is easy to see that

$$k\aleph_0 = \aleph_0$$

where k is ANY finite number.
At this point it is quite natural to ask:
"How about $\aleph_0 \cdot \aleph_0$?"
That is,
what do you get when
you MULTIPLY
one denumerable set by another?
Do you still get $\aleph_0$,
or something larger?

To answer this
consider the following ingenious idea!
First of all,
please remember that

when, in ordinary algebra,
you multiply

$(a + b + c)$ by $(d + e)$

you get

$ad + bd + cd + ae + be + ce$.

That is,
each term in the multiplicand
is "associated" with
each term in the multiplier,
thus obtaining
the totality of all such pairs.
Now here comes
the ingenious idea referred to above:
To associate each term of the first $\aleph_0$,
with each term of the second $\aleph_0$,
arrange the terms of the first set
along the top row,
and the terms of the second set
down the first column
of the following table:*

	1	2	3	4	5	....
1	1	2	4	7	11	....
2	3	5	8	12		
3	6	9	13			
4	10	14				
5	15					
.	.					
.	.					
.	.					
.	.					
.	.					

---

*See "Actual Infinity" by
S. A. Bogomolov (1934), p. 16.

and now number the boxes
in the body of the table,
in order,
1, 2, 3, ...
along the diagonals as shown.
If you now take
any term of the first set
(in the top row),
say 4,
and any term of the second set
(in the first column),
say 2,
there is "associated" with this pair,
the definite box 12,
just as you would
"look up" the result in
any "multiplication table".
And conversely,
each box,
say the one numbered 9
is "associated" with the pair:
the term 2 of the first set (top row)
and the term 3 of the
second set (first column).
Thus the TOTALITY of such pairs*
is represented by
the numbered boxes,

---

*This definition of "multiplication" of
two sets
applies to sets that have
no elements in common
(as in the two algebraic expressions
on p. 119),
and must be distinguished from the
more general definition
of the "product" of two classes,
meaning that part which
IS COMMON to both,
as in Boolean Algebra.
See my little book:
"Mits, Wits, and Logic".

and this set of boxes,
is obviously "countable",
for they were numbered in order
1, 2, 3, 4, 5, ... (see p. 120)
in the first place.
Hence we see that

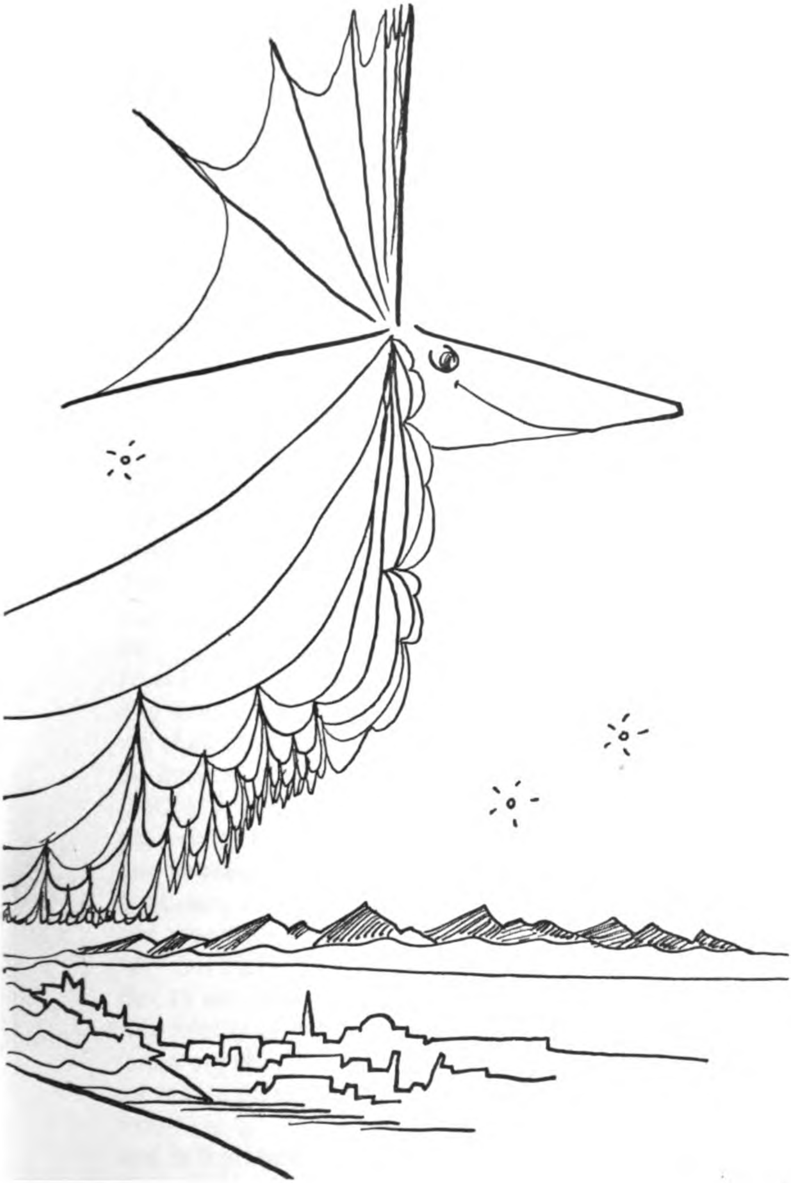
$$\aleph_0 \cdot \aleph_0 = \aleph_0.$$

And perhaps you are wondering
whether there is ANY operation which
when performed on $\aleph_0$'s
leads to a transfinite GREATER than $\aleph_0$?

The answer is
"YES",
as you will see from
a simple and interesting illustration
in the next chapter.

I hope you are
not getting tired of
these details,
but see in them
the INVENTIVENESS of the HUMAN mind.
And
if you are the "practical" type,
you will have more respect
for all this
when you realize later
that it has
most important practical value.

But let us not wait for
these practical applications,
and in the meantime be bored until
we see the gadgets emerge!
For although the gadgets



are important enough,
you must admit that
a realization of
WHAT WE HUMANS ARE REALLY LIKE,
what our minds are CAPABLE of doing, —
is so much MORE important,
for the gadgets alone
may actually kill us,
whereas
the flowering of
the BEST that is in HUMAN NATURE,
the SAM IN US,
will give us
a new lease on
LIFE and HAPPINESS.

10 How To Go from $\aleph_0$ to c

Consider a real number
of the form $0.a_1$
where a_1 may be replaced by
any of the 10 digits:
0, 1, 2, 3, 4, 5, 6, 7, 8, 9.
We thus have 10 possible values for $0.a_1$.
Similarly $0.a_1a_2$,
in which EACH a may be replaced by
any one of the 10 digits,
can represent 10×10 or 10^2
DIFFERENT numbers.
And $0.a_1a_2a_3$
can represent 10^3 DIFFERENT numbers
etc.
Now if we take $a_1a_2a_3a_4 \ldots\ldots$
and keep on going,
we shall obviously have
$\aleph_0$ decimal places,
EACH of which may be filled in by
any one of the 10 digits from 0 to 9,
and consequently
$0.a_1a_2a_3a_4 \ldots\ldots$
can represent
$10^{\aleph_0}$ DIFFERENT numbers.
But as was stated on p. 113
this totality of numbers
is the set of
all real numbers in the interval
from 0 to 1,
and is therefore equal to c (p. 116),

which is also
the "transfinite number" giving
the "power" of
ALL real numbers (p. 117)
(positive, negative, and zero).
Thus

$$10^{\aleph_0} = c.$$

And it can easily be shown that

$$n^{\aleph_0} = c$$

where n is ANY finite number.
Let me show this for the case where

$$n = 2,$$

because it will throw light on
our wonderful "place" system of
writing numbers
(which we use daily, but rarely stop to
consider how wonderful it is),
as well as on other possible
and very practical systems.

In our ordinary system,
based on the "radix" 10,
you know that

325

means 3 × 100 + 2 × 10 + 5;
that is,
the 3 stands for 300 (or 3 × 100)
since it is located in
"hundred's" place,
the 2 stands for 20 (or 2 × 10)
since it is in "ten's" place,
and the 5 is 5 units
since it is in "unit's" place.

Or, if other digits are located in
these various places,
we have

$$u + 10t + 100h + \text{etc.},$$

or

$$u + 10t + 10^2h + \text{etc.},$$

where u, t, h, etc., are
the digits in the
unit's, ten's, hundred's, etc. places,
and each may have a value of
any digit from 0 to 9, inclusive,
as you know perfectly well.

And you know how easy it is,
with this place system,
to add numbers,
to multiply them,
to write any number we require,
no matter how large it may be.

Stop to consider
how difficult all these things
would become if we had stuck to
the old Roman number system,
in which the number 1951
must be written thus:

MCMLI

as is sometimes done when
"laying a cornerstone"
of a building.
Such a number is difficult
even to READ!
But have you ever tried
to ADD two of them together,
or to MULTIPLY them?

If you have,
you will not only be glad that
this system was discarded,
but you will appreciate
the one we use
more than you ordinarily do.
As with other blessings,
we are apt to ignore them
unless we are made
CONSCIOUS of them![*]

But our common system,
based on 10,
is NOT the ONLY
good practical system.
Thus one in common use in
the modern electronic
calculating machines
is based on the "radix" 2,
instead of 10.
Thus, instead of writing a number as
$u + 10t + 10^2h +$ etc.
in which 10 is the base,
it can also be written as
$a + 2b + 2^2c +$ etc.
where the base, or radix, is now
2 instead of 10,
and the a, b, c, etc.
may now each have only
the two values 0 or 1
(since these are the only digits
which are less than the radix 2,
just as, when using 10 as a radix,

---

[*] See the clever device used by
Philip Wylie in his novel
"Disappearance", in which he makes
men and women CONSCIOUS of
the blessing of having each other!

we could write any digit less than 10
in any of the "places").

Compare the following numbers
written in the two systems:

radix 10		radix 2	
8	⟶	1000	(since this means $1(2)^3 + 0(2)^2 + 0(2) + 0$)
5	⟶	101	(that is, $1(2)^2 + 0(2) + 1$)
etc.		etc.	

When adding two numbers in
the "binary" system (radix 2),
as soon as a column adds up to
2 or more,
you "carry" multiples of powers of 2
to the next column,
just as in the decimal system (radix 10)
you "carry" over any amount which is
a multiple of 10, 10^2, 10^3, etc.
in the various columns.
Thus, in the binary system
the sum of 100 and 101 is:

$$\begin{array}{rl} 100 & (= 1(2)^2 + 0(2) + 0) \\ 101 & (= 1(2)^2 + 0(2) + 1) \\ \hline 1001 & (= 1(2)^3 + 0(2)^2 + 0(2) + 1) \end{array}$$

which, in the decimal system,
would be equivalent to
the sum of 4 and 5,
giving the result, 9.
The reason that this binary system is
so useful in electronic calculators
is that
since there are only
two alternatives, 0 or 1,

in each place,
an electronic tube in a given place
can easily register
the two alternatives by
having the tube "on" or "off".
Perhaps this system
may not appeal to you
because it is unfamiliar to you.
But
its ADVANTAGE would be
that it would be much
EASIER to ADD numbers
(since there are only
two possible digits, 0 and 1,
and you would not have to
learn combinations like
$8 + 3$, $6 + 2$,
and many, many others).
On the other hand,
the DISADVANTAGE would be
that it takes many more "places"
to represent a number in the
binary system than it does in the
decimal system:
thus,
our ordinary 9 becomes 1001
(that is $1(2)^3 + 0(2)^2 + 0(2) + 1$.
And, of course,
larger numbers would take
an ENORMOUS number of places!
But in the machines,
after building in
many, many electronic tubes,
it is then quite simple
for the machine to
flash each tube "on" or "off",
since only two alternative signals

are needed in each place,
as mentioned above.

Now, similarly,
real numbers between 0 and 1
may be expressed in
either the decimal or the binary system,
thus:
0.1 in the decimal system means 1/10
whereas
0.1 in the binary system means 1/2.
Similarly
0.09 in the decimal system means

$$0/10 + 9/100$$

or

$$0/10 + 9/10^2$$

etc.
Whereas,
in the binary system,
each place could have
ONLY a 0 or a 1 in it,
so that
0.101 would mean $1/2 + 0/2^2 + 1/2^3$
which is equivalent to
$1/2 + 1/8 = 5/8$
and 5/8 in the decimal system
becomes 0.625
which means $6/10 + 2/100 + 5/1000$
or

$$\frac{600 + 20 + 5}{1000} = \frac{625}{1000} = 0.625.$$

Thus the DECIMAL 0.625
has the same value as
0.101 in the BINARY system,
etc., etc.

You need not worry
about using the binary system,
for it would naturally take
a little time
for you to get accustomed to it.
But it is important to know

(1) that systems other than
the decimal one
are possible;

(2) that some of these other systems
may even be PREFERABLE
for some purposes,
as in electronic calculators.

And now let us go back to p. 126,
where all this started,
to apply it quite simply
to show that

$$2^{\aleph_0} = c.$$

For,
you can now see that
any real number in the interval
from 0 to 1,
may be written in the binary system
thus:

$0.\varepsilon_1\varepsilon_2\varepsilon_3 \ldots\ldots$

where each ε may have only
one of the two possible values 0 or 1
(instead of the 10 possibilities
from 0 to 9, as in the
decimal system, — see p. 113).
And so
$0.\varepsilon_1$ can represent only
0.0 or 0.1

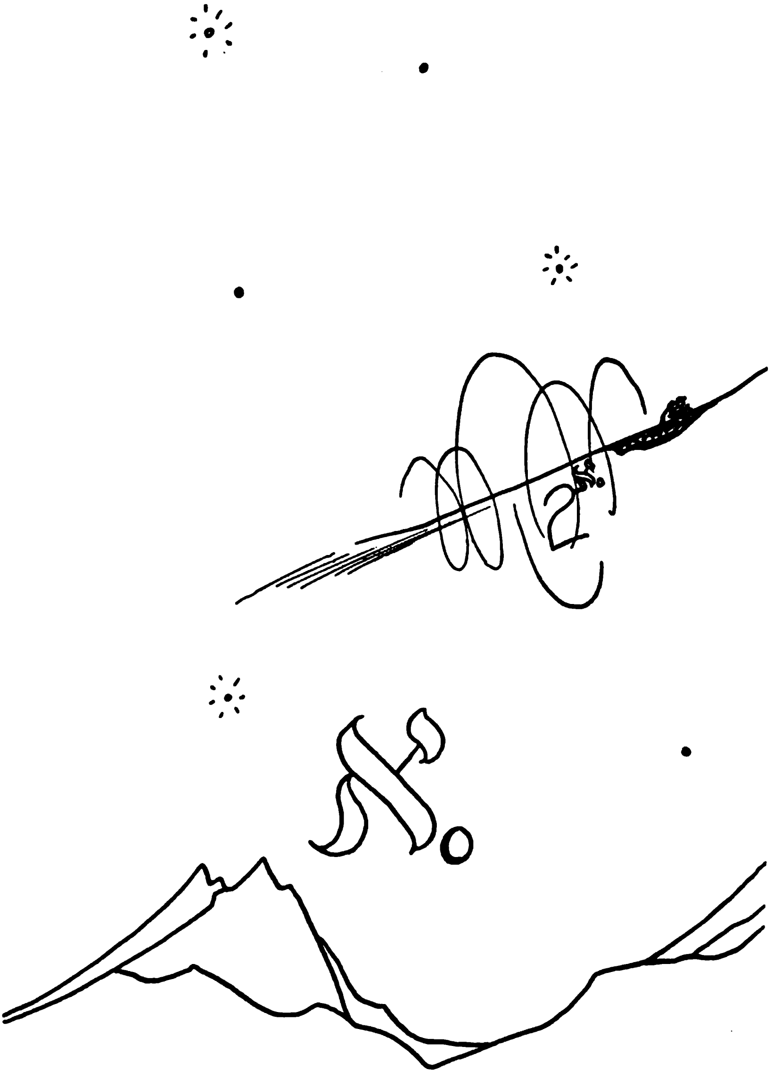
(i.e. 2 different numbers only);
similarly
$0.\varepsilon_1\varepsilon_2$ can represent only
0.00 or 0.01 or 0.10 or 0.11
(i.e. 4 different numbers only).
And similarly
$0.\varepsilon_1\varepsilon_2\varepsilon_3$ can represent
$2 \times 2 \times 2$ or 2^3
different numbers.
And consequently
if there are $\aleph_0$ different "places"
(as is needed for each real number in
this interval —
see p. 113)
there would be $2^{\aleph_0}$ possible
different real numbers in
this interval.
But this totality has already
been shown to have the "power" c
(see p. 115),
hence

$$2^{\aleph_0} = c$$

as was promised on p. 132.
And similarly for
any finite number $\geqq 2$*
as "radix".

---

*The symbol $\geqq$ means (as you probably know)
"greater than or equal to".
Hence the above statement in the text
means that the "radix" may be 2 or
any LARGER finite integer.
Note that 1 cannot be used as a radix
for a number system, since
1, 1^2, 1^3, etc. are each equal to 1,
so that there would be no distinction
between the "places"
as we get from 10, 10^2, 10^3 etc.
(ten's place, hundred's place, etc.)
or from 2, 2^2, 2^3, etc., or from
any radix $\geqq 2$.


$2^{\aleph_0}$
$\aleph_0$

To sum up then:

(1) $n\aleph_0 = \aleph_0$
that is:
a denumerable infinity
multiplied by
any finite number
gives a denumerable infinity (p. 118).

(2) $\aleph_0 \cdot \aleph_0 = \aleph_0{}^2 = \aleph_0$ (p. 121)

and hence

(3) $\aleph_0^n = \aleph_0$
that is
a denumerable infinity raised to
any finite power
still gives
a denumerable infinity.

(4) But $n^{\aleph_0} = c$ (p. 126)
i.e. any finite number $\geqq 2$
when raised to a power of
a denumerable infinity
gives c
which is a GREATER transfinite
than $\aleph_0$ (p. 115)!

From this will follow
some amazing consequences,
as you will see in
the next three chapters.

11 Operations on Transfinites

You know that
when you play any game
you must know the rules of
the game in question.
And when you "play the game" of
ordinary arithmetic or algebra* or
other algebras* or
one of the geometries (see Ch. 5),
etc., etc.
you must also know
the basic rules or
POSTULATES of
the branch of mathematics in question.
And of course the same applies to
Cantor's Theory of Transfinites.
I shall not give these postulates
to you here‡
but shall indicate
a few of the theorems,♦
so that you may see
how to operate with
transfinite numbers.

---

*See "The Fundamental Propositions of
Algebra" by E. V. Huntington,
a delightful little book,
published by
The Galois Institute Press of
Long Island University.

‡See "Zehn Vorlesungen uber
die Grundlegung der Mengenlehre"
by Adolf Fraenkel (1927).

♦See "A Survey of Modern Algebra" by
Birkhoff and MacLane.

As in ordinary arithmetic,
numbers are of two kinds:
cardinals and ordinals (see p. 163).
So also
in the arithmetic of transfinites,
there are cardinals and ordinals.
The transfinites you have already met,
$\aleph_0$ and c,
are CARDINAL transfinite numbers;
later (in Ch. 12)
you will learn something about
the transfinite ORDINALS.
But, for the present,
let me give you
a few of the
"rules" for transfinite CARDINALS:
if a, b, c, etc. represent
transfinite cardinals,
then

(1) $a + b = b + a$

(2) $a + (b + c) = (a + b) + c$

(3) $ab = ba$

(4) $a(bc) = (ab)c$

(5) $a(b + c) = ab + ac$

etc.

Now,
you know perfectly well
that
these rules apply also
in ordinary algebra and
arithmetic,
in which, however,

the letters a, b, c, etc.
represent ordinary finite numbers.
Similarly,
the following operations in
ordinary algebra* and arithmetic
apply also in
the arithmetic of transfinites:

(6) $a^m a^n = a^{m+n}$

(7) $(a^m)^n = a^{mn}$

(8) $(ab)^m = a^m b^m$

From all this
you might think that
the basic rules of
ordinary arithmetic and
the arithmetic of
transfinite cardinals
are exactly the same.

But do not be too hasty!

For here, for example,
is one DIFFERENCE between them:

the "cancellation laws"
which do hold in
ordinary arithmetic
DO NOT HOLD in
the arithmetic of
transfinite cardinals:
thus,

if $a + x = a + y$ (1)

---

*Look this up in
any book on elementary algebra.

then
in ordinary arithmetic
we may cancel the a's
(i.e. subtract a from
each side of equation (1))
and conclude that
equation (1) leads to

$$x = y. \tag{2}$$

But in the
arithmetic of transfinite numbers
equation (1) does NOT lead to (2):
for,

$$\left.\begin{array}{ll}\text{since} & 1 + \aleph_0 = \aleph_0 \\ \text{and} & 2 + \aleph_0 = \aleph_0\end{array}\right\} \text{(p. 117)}$$

$$\text{hence} \quad 1 + \aleph_0 = 2 + \aleph_0;$$

but you can see that
cancelling the $\aleph_0$'s
CANNOT BE ALLOWED
for then you would get

$$1 = 2$$

which would be
INTOLERABLE,
would it not?!
For if you had two dollars and
you were given
its EQUAL (!), one dollar,
in place of your
two dollars,
you would suspect skulduggery
would you not?

And similarly

$$\text{if } ax = ay \tag{3}$$

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then you may cancel the a's
(i.e. divide both sides of (3) by a
if $a \neq 0$)
in ORDINARY arithmetic
and thus get

$$x = y \tag{4}$$

BUT
not so with transfinites
where

$$\left.\begin{aligned} 2\aleph_0 &= \aleph_0 \\ \text{and} \quad 3\aleph_0 &= \aleph_0 \end{aligned}\right\} \text{(p. 118)}$$

and hence

$$2\aleph_0 = 3\aleph_0$$

but you MUST NOT cancel the $\aleph_0$'s
for that would give

$$2 = 3.$$

Later (p. 167) you will see that
transfinite ORDINALS
are NOT governed by
the same rules as
transfinite CARDINALS —
for even (1) on p. 138
does NOT apply to the
ORDINALS!

And now you can appreciate
some amazing consequences:

since $$2^{\aleph_0} = c \text{ (p. 133)},$$

consequently

$$cc = 2^{\aleph_0} \cdot 2^{\aleph_0} = 2^{2\aleph_0} = 2^{\aleph_0} = c \text{ (pp. 118 and 139)},$$

and thus

$$cc = c.$$

Hence

$$ccc = (cc)c = cc = c$$

or

$$c^3 = c,$$

and, in general,

$$c^n = c$$

(where n is any positive
finite integer),
which is a most remarkable idea
as you can see if
you stop to consider its
MEANING:

For, take the simple case where $n = 2$,
then $c^2 = c$, as was shown above;
and then
since c is the "power" of
the set of real points on
the line-segment from 0 to 1 (p. 125),
then $c^2 = c$ means that
the "number" of real points in
a unit SQUARE (c^2) is
the SAME as
the "number" of real points in
one of its sides (c)!
Although this was proved on p. 141,
it still seems a most strange idea,
does it not?
But consider the following graph
(on p. 143):
Suppose A is a point somewhere
within the unit square, as shown.
This point, A,
has two coordinates, x and y,
both of these being real numbers
having values between 0 and 1.

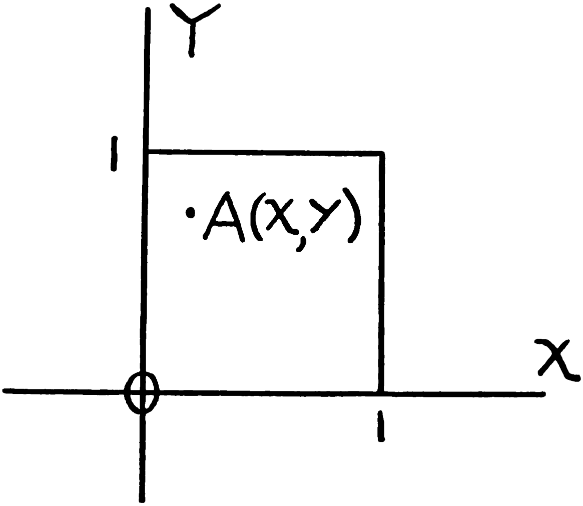
Suppose that x is

$$0.a_1a_2a_3a_4 \ldots\ldots, \quad (5)$$

some real number
(rational or irrational)
between 0 and 1,
and that y is

$$0.b_1b_2b_3b_4 \ldots\ldots, \quad (6)$$

another real number
(rational or irrational)
between 0 and 1.



Now obviously,
from both (5) and (6) above
it is possible to form
one SINGLE real number
thus:
$0.a_1b_1a_2b_2a_3b_3a_4b_4 \ldots\ldots\ldots,$
and this resulting number
also being between 0 and 1,

will be located somewhere on
the X-axis between 0 and 1,
will it not?
And similarly
any real number between 0 and 1
on the X-axis,
like $0.c_1d_1c_2d_2c_3d_3c_4d_4$ . . . .
may be split into TWO numbers,
$0.c_1c_2c_3c_4$ . . . . and $d_1d_2d_3d_4$ . . . .
which may serve as
the coordinates of
some point in the square on p. 143.

And so there is a
1-1 correspondence between
the points on the LINE-segment
on the X-axis, from 0 to 1,
and the points in the SQUARE,
so that these two sets of points are
equivalent.
I hope that the equation $c^2 = c$,
after the proof on p. 141 and
the discussion above,
is more acceptable to you now,
in spite of the fact that
it seemed strange to your "common sense".
For we MUST remember that
"common sense"
is really only a part of
the SAM within us,
AND that
SAM is a GROWING boy!
If we put him in a straitjacket,
by sticking to "common sense",
we shall stunt his growth and
keep him (and ourselves)
INFANTILE.

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Thus new ideas,
IF CONSISTENT,
are not really contrary to
"common sense",
but rather,
permit our "common sense" to GROW!

And similarly
even the more general case

$$c^n = c \text{ (p. 142)}$$

is entirely reasonable
and means that
the "number" of real points
not only in a square (c^2)
but also in a cube (c^3)
and even in an n-dimensional continuum
(where n is any positive integer)
has the same "power" as
a ONE-dimensional continuum!

Remember that
the RATIONAL numbers alone,
when placed on a line,
still left "gaps" on the line (p. 100)
but the REAL numbers
fill the line completely (p. 104)
and thus constitute a
"continuous" set of points
or a "continuum".
This of course applies
not only on a line
(which is a "one-dimensional continuum")
but similarly
a square is a "two-dimensional continuum"
if it has no "gaps",
and so on for higher dimensions.

Perhaps you are troubled by
the idea of "higher" dimensions —
you may go along as far as
THREE dimensions, a cube,
but what about the "fourth dimension",
and the fifth and sixth and so on?

Is your "common sense" troubling you again,
and do you need SAM'S help again? —

if so,
it is entirely natural and normal,
but let us not just leave it at that
but rather GET the needed help from
SAM
who is fast outgrowing that
straitjacket of "common sense".
So let me say something about
"higher" dimensions.

To make it clear,
permit me to tell you
a personal experience.

12 Higher Dimensions

Not long ago,
a gentleman who used to teach English,
suddenly became enthusiastic about
mathematics,
and decided that he would rather
teach that subject,
for he felt that
mathematics is very poorly taught
(he certainly had a point there! —
though probably
the same may be said of
all subjects,
for education is still in
a most deplorable condition
on nearly ALL fronts!) ;
he felt that he could do a better job by
making it more "visual" and
less "abstract",
and hence more acceptable to
that old "common sense",
and hence easier to understand.
Well and good.
So he actually got himself a job
to teach mathematics in a high school,
for he was in dead earnest.
Now let me give you
one illustration of
his procedure —
you will find it to be
both instructive and amusing,
and it will throw light on
the tremendous VALUE of the ABSTRACT,

a very PRACTICAL value!

One of the theorems that
many students find difficulty with
is the "binomial theorem":
a "binomial" is
an algebraic expression having
exactly TWO terms, like

$a + b$, or $x - 2y$, or $3m/n + 5z$,

etc.*
Now you know that
$(a + b)^2$ means $(a + b)(a + b)$
and, to find this product
a student can of course "multiply it out"!
But the "binomial theorem"
(originated by Newton
when he was still an undergraduate)
gives a way of
arriving at the product
WITHOUT actually multiplying
$(a + b)$ by $(a + b)$,
and it turns out that
the answer you get (by either method)
is $a^2 + 2ab + b^2$.
If a student gets this answer by
actual multiplication,
he does not worry about it,

---

*Note that the two "terms" are
separated by a + or − sign,
but that WITHIN each term
there are NO + or − signs.
This is only a crude definition
of a binomial,
but I hope it will suffice
for the moment.
This is not the place to
go into TOO many details.
But you should really look this up
in a good book on elementary algebra.

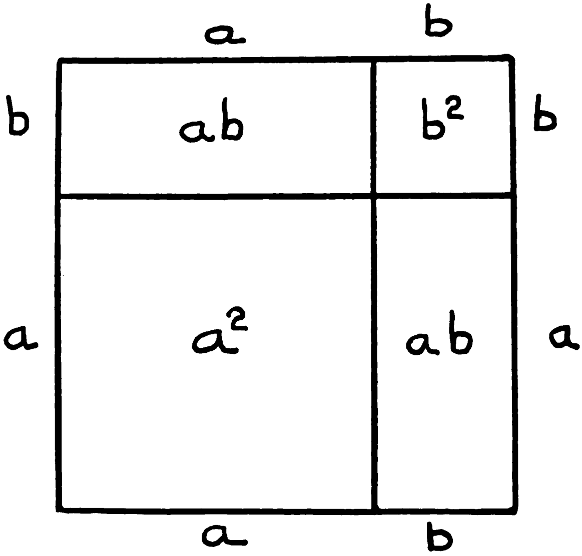
for he learns to do it
quite mechanically.
But when he is asked to find
$(a + b)^2$ by means of
the binomial theorem,
he is very likely to give as the answer,
just $a^2 + b^2$,
and, when told that
he has forgotten to put in
the middle term, 2ab,
he often does not "understand",
for it seems to him
so "reasonable"
(whatever that means!)
that $(a + b)^2$ should equal $a^2 + b^2$
that he wonders
"Where the heck does that
2ab come from?"
Please do not think that
I am implying
that "reasonable" does not
mean anything!
I am saying only that
his use of this word is meaningless.
What "reasonable" means,
in mathematics,
is to start with an adequate
set of postulates (basic rules) and
derive from it,
by means of an adequate logic,*
the consequences or theorems that
follow from the postulates.

---

*The very idea that there are
different logics
(just as there are
different algebras and
different geometries — see p. 137)
may seem strange to you,
but see p. 336.

When you do this
you are “thinking”,
you are being “reasonable”,
as these words are used by SAM!

Draw a square whose length and width
are each $(a + b)$ as shown:



Now $(a + b)^2$ represents
the AREA of this square, does it not?
And, as you see,
the area is made up of
FOUR parts:
two of these parts are squares,
whose areas are, respectively,
a^2 and b^2;
AND there are two more parts,
rectangles,
the dimensions of each being a and b

and hence
the AREA of EACH is ab,
so that the sum of these two areas is 2ab.
Finally,
adding all four parts together,
we then get

$$a^2 + 2ab + b^2$$

as the area of the original square,
or $(a + b)^2$.
And you will agree that
this geometric approach
DOES help to see
where the 2ab comes from
and that the young student
IS less likely to forget
the correct result,
after seeing it in this
graphic and convincing way!

And then naturally
the literary gentleman
decided to make clear,
in a similar manner,
why it is that
$(a + b)^3$ is equal to
$a^3 + 3a^2b + 3ab^2 + b^3$,
for this is the result obtained
either by multiplying
$(a + b)$ by $(a + b)$ by $(a + b)$
or by means of the
"binomial theorem"
(which, by the way, shows
how to find $(a + b)^n$,
where n is ANY positive integer*.)

---

*The binomial theorem has been
further developed

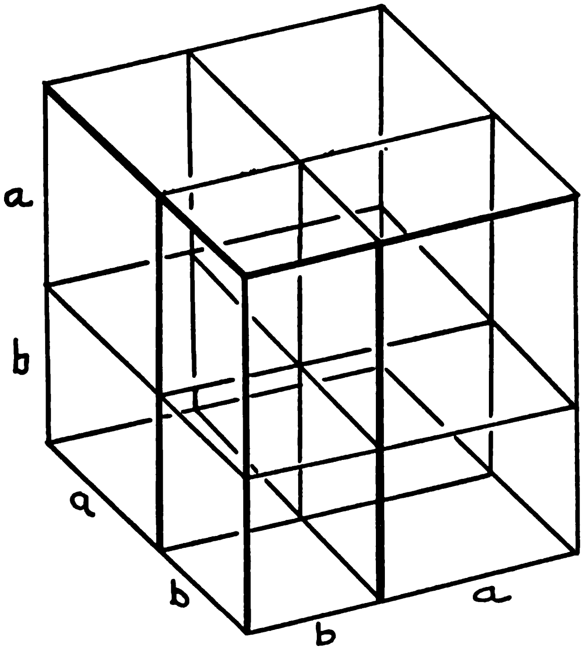
Now, of course,
multiplying this out
gets to be pretty tedious,
so the average student
is not over anxious to use this method;
but
when he considers using
the binomial theorem,
the result seems
so "unreasonable" again,
that he begins to feel surrounded
and often thinks,
"Well, this is the
last course in algebra that
I will take.
This stuff don't make no sense!"
So our literary gentleman
wanted to come to his rescue —
a most worthy cause,
you will admit!
And he proceeded as follows:

He asked his wife
(who is most cooperative and kindly)
to bake a cake,
with frosting and everything,
in the shape of a cube.
He then brought

---

so that n may be
not only any positive integer,
but also any rational number,
or any real number,
or even any "complex number"!
Please look this up,
in a book on the theory of functions
of a complex variable;
it is very interesting in itself,
but naturally we cannot
take all detours
on this trip here.

this delicious cake
to his class
and proceeded to cut it up
like this
(as shown
in the following diagram):



he marked a point on
the length, the width, and the height
on the cake,
splitting each of these dimensions
into two parts, a and b, as shown,
(as he had done with
the square on p. 150),
and then cut the cube at these points,
and actually showed that

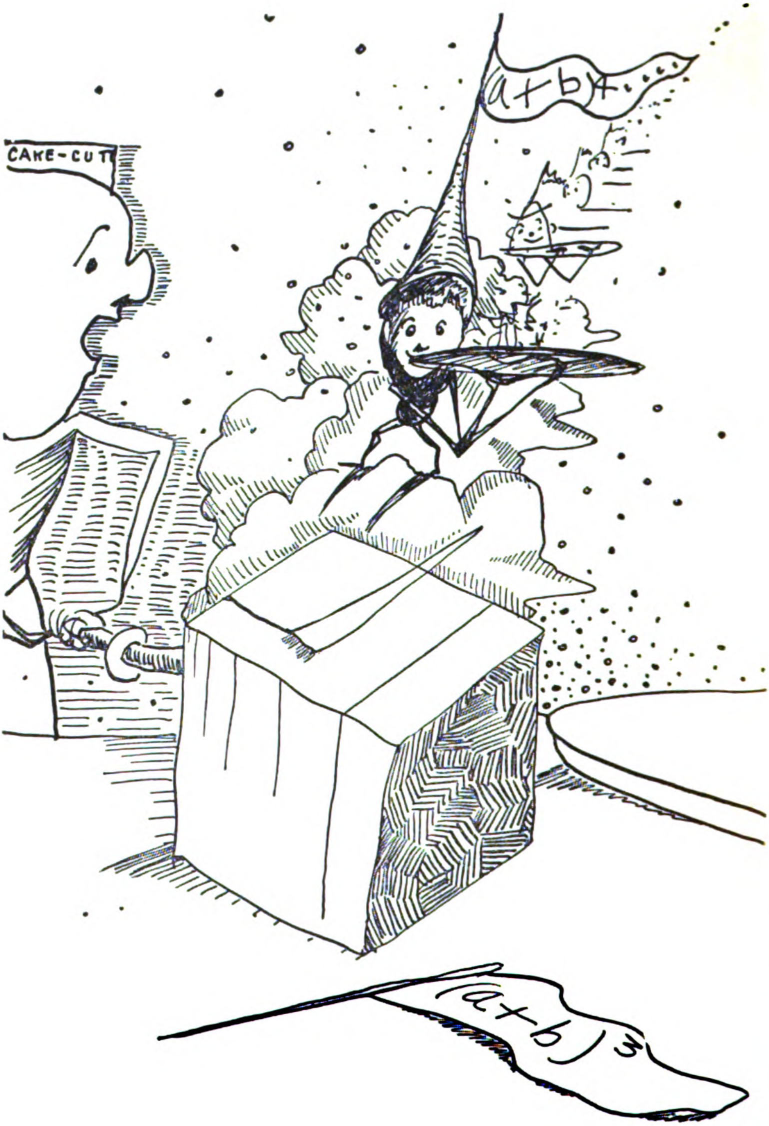
the cake was thus cut up into
the following EIGHT pieces:
in one corner
he got a piece the size of which is a^3,
in another corner he got b^3,
then he got three pieces each of which
has length a, width a, and height b,
and therefore the volume of
EACH of THESE
is a^2b, and
the three together are $3a^2b$;
and finally he got
three more pieces EACH of which
has length a, width b, height b,
so that the volume of each of these
is ab^2, and
these three together make $3ab^2$;
and thus the whole cake,
consisting of these eight pieces,
obviously has a TOTAL volume of

$$a^3 + 3a^2b + 3ab^2 + b^3.$$

Now perhaps you think that
this is not at all clear,
and that you would even prefer
to multiply
$(a + b)$ by $(a + b)$ by $(a + b)$,
tedious as this may be,
since the "cutting up" may seem even
more complicated to you.
But let us be fair!
For my description in words,
even with the aid of
the diagram on p. 153,
is certainly not to be compared with
that cake-party
the students were treated to!

For not only did they
enjoy eating the cake after
the "cutting up",
but they actually
saw with their own eyes
the eight pieces mentioned above!
And so,
let us be generous
and give the gentleman
the benefit of the doubt,
and believe that it
really helped the students to see why
$(a + b)^3$ is equal to
$a^3 + 3a^2b + 3ab^2 + b^3$.
If you are skeptical,
bake yourself a cake and try it.
Let me know
how the whole thing turns out.

But now
I am coming to the
real moral of this story.
For,
granting everything so far,
let us now ask what
the literary gentleman would do with
$(a + b)^4$?
For the best of wives
cannot bake him a
FOUR-dimensional cake!
And the multiplication process is now
more tedious than ever,
for you now have to multiply
$(a + b)$ by $(a + b)$ by $(a + b)$ by $(a + b)$,
there are now FOUR of them to multiply.
Thus now the binomial theorem
comes to your rescue in real earnest,


CAKE-CUT
$(a+b)^4 \ldots$
$(a+b)^3$

for, by means of it,
you can immediately write the result:
$a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, —
if, of course,
you have learned this theorem,
which is really quite easy!
Perhaps you may still say:
"But why bother?
Who could ever possibly
want to find $(a + b)^4$?
Why bother about the whole thing?"

Well, Sir, or Madam,
if you ever put any money into a bank,
at compound interest, —
and I hope you will —
and if you should want to know
how much your original principal
would amount to in four years,
this is the very formula
you would need!
And of course,
the binomial theorem
would answer your question
for any number of years,
whether compounded annually or
semi-annually or
quarterly or
for any other interest period.
And, what is more,
the same binomial theorem
would help you also
in figuring out
annuity payments to which
you might be entitled, and
the amounts you should pay if
you are buying anything on

the installment plan,
and many, many other things.
Of course you might say
"But why not 'Let George do it'
and forget the whole thing?!"
But,
if we are going to take the attitude
"Let George do it"
all along the line,
we shall certainly find ourselves
swindled right and left —
if we do not do at least
SOME of our own thinking
in SOME field.
If we are going to let "George"
figure out for us
who are to be our candidates for
all kinds of important positions,
and let "George" do our voting for us,
and make all our decisions for us —
but why do I say "if",
isn't this what
many, many of us really do?
And isn't this why the world is
in such a mess?
Is it not
the indifference and the apathy
of so many of us
that gives such opportunities to
anyone who is
"smart" enough to swindle us
out of house and home?
Has it not become
IMPERATIVE
for us to do some of our own
"figuring" out
or "reasoning"

or “thinking”?
And if so,
has it not become
IMPERATIVE
to know what “reasoning” means?
Or else how can we do it?
For the average use of words like
“reasoning”,
“logic”,
“consistency”, etc., etc.,
is so APPALLINGLY inadequate,
even when used by
the best-natured “cake-cutters”,
not to mention the “smart”
lunatic swindlers in our midst —
well, I say,
we must PROTECT ourselves
by getting some things STRAIGHT,
VERY straight,
and CLEAR.
And SAM will help us if
we give him even
half a chance.

It will take some effort,
but not nearly as much effort as
will be necessary to
clean up the mess if
we just let things drift and
“Let George do it”.

And what has the study of
INFINITY
to do with all this?
Well,
it is a beautiful illustration of
HOW to THINK,

and this is by far
the most IMPORTANT thing
for us to learn,
and the most enjoyable too —
for it lifts us out of
the horrible "boredom"
which results from that old
"Let George do it"!

13 A Hierarchy of Infinities!

Before going into this Hierarchy,
permit me to say a few more words
about "higher" dimensions.
On p. 142 you saw that

$$c^n = c,$$

where n may be
any finite positive integer,
and hence that
even an n-dimensional continuum
has the same number of points as
a one-dimensional continuum.
But this is not all!

For,
as you already know, $c = 2^{\aleph_0}$.
Hence $c^{\aleph_0} = (2^{\aleph_0})^{\aleph_0}$
or $c^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = c$ (pp. 121 and 139).
Thus $c^{\aleph_0} = c$,
which says that a continuum of
a denumerable INFINITY of dimensions
still has the same "power" as
a ONE-dimensional continuum!
You must agree that this is a truly
GRAND THEOREM —
and that your past worries about
the "FOURTH dimension"
seem rather tame when you consider
a denumerable INFINITY of dimensions!
And yet the procedures described above

are no harder to follow than those of
ordinary elementary algebra.
Is it not a wonderful fact that
the human mind
is capable of so much
IMAGINATION
which is so
FERTILE!
HOW fertile and USEFUL is
Cantor's Theory
you have yet to see.
But perhaps you can already appreciate
the tremendous IMAGINATION
that was required to build up
this Theory of Transfinites,
and can thus already appreciate
the remark about Cantor's work
made by Hilbert (see p. 106).
For here,
as in other works of SAM,
you see the human spirit in its
naked beauty and dignity,
just as the human body is seen in its
naked beauty and dignity
in the classical Greek statues.
And we then realize how inadequate is
the "education" which covers up
this natural beauty of
spirit and body
with layer upon layer of
"boloney" skin.*

---

*No wonder that so many cynics say that
if "you scratch the surface of
a human being,
you find a 'beast',
full of hatred and murderous thoughts!"
The fact is that if you merely
scratch the surface, you reach only
some more putrid layers of

In order to see how
Cantor built up
many infinities, without end,
let us go back for a moment
to ordinary finite integers.
These are of two kinds:
cardinals and ordinals.
CARDINAL NUMBERS are
1, 2, 3, 4, ... ;
ORDINAL NUMBERS are
1st, 2nd, 3rd, 4th, ...
Thus if you have
a set of objects,
and if the ORDER of their arrangement
is NOT significant,
you designate their totality by
a CARDINAL number,
such as 5.
But if the order IS significant,
then you use ORDINAL numbers.
Now Cantor designated
the ORDINAL number of a set M
by $\overline{M}$,
the bar over the M showing that
the NATURE of the objects
is negated,
that is,
in merely COUNTING them,
it does not matter what they ARE;
and he designated the
CARDINAL number of the set M
by $\overline{\overline{M}}$,

---

the "bolony" skin in which
we have been wrapped.
One would have to remove
ALL that horrible accumulation
before reaching
the beautiful HEART of SAM.

the two bars over the M indicating
that NEITHER their ORDER
NOR their NATURE
is under consideration:
each bar over the M
designating a negation —
one bar showing the negation of the ORDER,
and the other bar showing the negation
of the NATURE of the objects.
And this applies
not only to finite sets
but also to infinite sets.
Thus,
$\aleph_0$ is a CARDINAL number,
representing, for example,
the totality of natural numbers
(meaning the positive integers),
1, 2, 3, 4, ...
And if you re-arranged them
in any way,
and "counted" them again,
you would of course
still get $\aleph_0$,
would you not?
You will remember that
in trying to "count"
the RATIONAL numbers,
it was permissible to
re-arrange them (see p. 94),
thus obtaining again
the CARDINAL number $\aleph_0$.
Whereas
the ORDINAL number of the set of
natural numbers,
is designated by the Greek letter ω.
And

any set which is "similar" to this one,
namely,
any set which has
the following properties:

(a) it has a FIRST element,
(b) each element in the set has
an IMMEDIATE SUCCESSOR,
(c) each element except the first
has an IMMEDIATE PREDECESSOR,
(d) it has NO LAST element —

any set having
these four properties
also has the ORDINAL number ω.
Further,
consider the set of
NEGATIVE integers:
$\ldots\ldots, -4, -3, -2, -1$
IN THIS ORDER.
This set has the properties:

(e) it has NO FIRST element,
(f) each element EXCEPT THE LAST
has an immediate successor,
(g) EACH ELEMENT has an
IMMEDIATE PREDECESSOR,

and

(h) it HAS a LAST element (namely, -1).
This set,
or any set "similar" to it,
is said to be
of order-type $^*\omega$ ("star ω"),
which is evidently different from ω,
as is indicated by

the difference between
properties (a) and (e),
as well as the difference between
(b) and (f),
etc.

Now if you consider
the set of numbers consisting of
all the negative integers and
zero and
all the positive integers,
in the order of magnitude,
thus
$\ldots\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\ldots$
such a set is designated by

$$^*\omega + \omega$$

or by the single letter π*
and is obviously NOT "similar" to
a set having order-type
either ω or $^*\omega$ —
note the requirements (a), (b), (c), (d)
for a set of order-type ω,
and the requirements (e), (f), (g), (h)
for a set of order-type $^*\omega$,
on p. 165.

Suppose now that you take
any set of order-type ω,
whose ordinal number is ω,
(and therefore having the properties
(a), (b), (c), (d) mentioned above)
and place in front of it

*This is of course NOT the π in
the formula for
the circumference of a circle:

$$c = 2\pi r.$$

a SINGLE new element,
its ordinal number would now be $1 + \omega$,
BUT this new set STILL has
the properties (a), (b), (c), (d),
does it not?
Hence $1 + \omega = \omega$.
On the other hand,
if you take the set of
natural numbers,
in their order of magnitude,
that is,
a set whose ordinal number is ω,
and place
a SINGLE new element AFTER it,
thus:

$$1, 2, 3, 4, \ldots, 1,$$

this new set
NO LONGER has the ordinal number ω
since this new set has
BOTH a FIRST AND a LAST element,
that is to say
it does NOT have property (d) on p. 165.
Its ordinal number is $\omega + 1$
and hence

$$\omega + 1 \neq \omega$$

(the symbol $\neq$ obviously means
"does not equal").
And so
since $1 + \omega = \omega$ (see above)
we see that

$$1 + \omega \neq \omega + 1.$$

Furthermore,
instead of adding a single element to ω,
we can also add
ANY finite number, k, of elements to ω,
obtaining either $k + \omega$ or $\omega + k$,

and here again it is obvious that
$k + \omega = \omega$ but $\omega + k \neq \omega$,
so that

$$k + \omega \neq \omega + k.$$

In other words,
addition here is
NOT commutative,
as it is in ordinary algebra
in which $a + b = b + a$
(for instance, $2 + 3 = 3 + 2$
since you get 5 either way),
where a and b represent
any two complex numbers.*
But of course
you are not surprised to find that
the basic postulates for
TRANSFINITE "numbers"
are DIFFERENT from
the basic postulates for
the numbers of ordinary algebra —
since each "system" has
its own characteristic set of
postulates (see p. 137).
Indeed you are already acquainted
with the idea that,

---

*A "complex" number is of the form
$x + iy$, where x and y are
real numbers, and $i = \sqrt{-1}$.
Of course if $y = 0$, then $x + iy = x$
which is a real number;
thus complex numbers include
real numbers.
Now, complex numbers are sufficient
for the solution of
any algebraic equation.
Thus complex numbers constitute
the "equipment" of
ordinary algebra.
See the little book by
E. V. Huntington,
referred to on p. 137.

for TRANSFINITES
the WHOLE is NOT NECESSARILY
greater than one of its PARTS
(Remember? see p. 85),
as is the case for
FINITE numbers.

Now an important concept which
will lead us to
a hierarchy of transfinites
is the concept of
a "WELL-ORDERED" set.
But first you must know
what a "SIMPLY-ORDERED" set is;
it is a set which has
the following three properties:

(a) If a and b are
distinct elements of the set,
then either a precedes b
or b precedes a,
for the elements are arranged in
some DEFINITE order;
(b) if a precedes b in the set
then a and b are distinct;
(c) if a precedes b and b precedes c,
then a precedes c in the set.

And a "WELL-ORDERED" set has,
in addition to these 3 properties,
also the following three properties:

(d) it HAS a FIRST element,
(e) each element EXCEPT the LAST,*

---

*You are surely not surprised that
an infinite set can have
BOTH a FIRST and a LAST element,
for consider the set $\omega + 1$ (p. 167).

IF THERE IS A LAST,
has an IMMEDIATE SUCCESSOR;
(f) every "fundamental segment" of the set
has a "limit":
a "fundamental segment" is
any lower segment which
has no last element;
the "limit" of
a fundamental segment is
the element
NEXT FOLLOWING
all the elements of
the fundamental segment.

For example,
in the set $\omega + 1$
1 is the limit of the
fundamental segment ω.

Thus a set of order-type $^*\omega$ (p. 165)
is SIMPLY-ORDERED but
NOT WELL-ORDERED
since it has NO FIRST element.
Also
the set of
POSITIVE RATIONAL NUMBERS
(which of course does NOT include 0
since 0 is NOT "positive"),
ARRANGED IN ORDER OF MAGNITUDE —
although it is a
SIMPLY-ORDERED set
yet it is NOT a WELL-ordered set
since it has NO FIRST element —
for, being a DENSE set (see p. 90),
there is no rational number
IMMEDIATELY after zero,
and, besides,

any rational number here
does NOT have
an IMMEDIATE successor
(since we are considering here
the set of rational numbers
arranged in
ORDER OF MAGNITUDE).
Such a set is designated by
the symbol η,
and its properties are:

(1) $\bar{\eta} = \aleph_0$,
(2) it has NO FIRST and NO LAST element,
(3) it is everywhere dense.

Furthermore,
the set of
RATIONAL NUMBERS from 0 to 1 INCLUSIVE
would have the order-type $1 + \eta + 1$.
But the set of
REAL NUMBERS from 0 to 1 INCLUSIVE
is obviously DIFFERENT from this
(it is designated by θ)
for, as you know, $\bar{\theta} \neq \aleph_0$
since the continuum of real numbers is
NOT a countable set (see Ch. 9).

Let us next examine
the sets $2 \cdot \omega$ and $\omega \cdot 2$:
the former is of the type
$a_1, b_1;\ a_2, b_2;\ a_3, b_3;\ \ldots .\ a_n, b_n;\ \ldots .$
that is,
since the multiplicand is 2,
and the multiplier is ω,
hence the product
consists of PAIRS of objects,
and the total set of PAIRS is

obviously
in 1-1 correspondence with
the natural numbers 1, 2, 3, 4, . . .
since there is a FIRST PAIR (a_1, b_1),
a SECOND PAIR (a_2, b_2), etc.
and therefore $2 \cdot \omega$ again has
the ORDINAL number ω.
BUT
$\omega \cdot 2$ is of the type
$a_1, a_2, a_3, \ldots, a_n, \ldots; \; b_1, b_2, b_3, \ldots, b_n, \ldots;$
and this set $\neq \omega$
since b_1 has
NO IMMEDIATE PREDECESSOR
(see property (c) on p. 165).
Thus $\omega \cdot 2 \neq \omega$.
Hence $2 \cdot \omega \neq \omega \cdot 2$.
And, in general,

$$k \cdot \omega \neq \omega \cdot k$$

so that
multiplication here, like addition (p. 168),
is also NOT COMMUTATIVE
(as it is in ordinary algebra in which $ab = ba$,
where a and b represent
any two complex numbers — see p. 168).

Note that all these sets
(on pp. 164 to 171)
even though they may be of
DIFFERENT order-types
(like ω and η, for instance)
MAY still have
the SAME CARDINAL number, $\aleph_0$,
IF they can be RE-ARRANGED so as to be
in 1-1 correspondence with
the natural numbers.
Thus even the set $\omega \cdot 2$ (above)

can be RE-ARRANGED thus:
$a_1, b_1, a_2, b_2, \ldots$
showing that it too is
a COUNTABLE infinity
and therefore has the "power" or
CARDINAL NUMBER, $\aleph_0$,
although its ORDINAL NUMBER,
in the order given on p. 172,
is NOT ω but $\omega + \omega$.

Now Cantor has shown how to construct
many, many ordinal numbers like
$\omega \cdot k + m$,[*] $\ldots\ldots, \omega^2, \ldots\ldots, \omega^2 + k, \ldots\ldots, \omega^2 + \omega, \ldots\ldots,$
$\omega^2 + \omega + k, \ldots\ldots, \omega^2 + \omega \cdot 2, \ldots\ldots \omega^2 + \omega \cdot k + m, \ldots\ldots,$
$\omega^2 \cdot 2, \ldots\ldots, \omega^2 \cdot k, \ldots\ldots, \omega^3, \ldots\ldots, \omega^k, \ldots\ldots, \omega^\omega, \ldots\ldots,$
$\omega^{\omega^\omega}, \ldots\ldots$
which are the ORDINAL NUMBERS of
WELL-ORDERED SETS
ALL of which have
the SAME CARDINAL NUMBER, $\aleph_0$,
since, to get the CARDINAL NUMBER
they can ALL be RE-ARRANGED so as to be
in 1-1 correspondence with
the natural numbers: 1, 2, 3, 4, . . .

---

[*] Thus if $k = 1$, and m increases to ω,
$\omega \cdot k + m$ becomes $\omega + \omega$,
for example the set:
1, 2, 3, 4, · · · , 1, 2, 3, 4, · · ·
which is obviously a well-ordered set
since it satisfies
requirements (d), (e), (f) on pp. 169 and 170,
although it is of course
NOT of the type ω(1, 2, 3, 4, · · ·)
since it has an "endless tail" WITHIN
the set
as well as at the END.
This is of course
the same as $\omega \cdot 2$ shown on p. 172.
And similarly for
1, 2, 3, 4, . . . , 1 2, 3, 4, . . . 1, 2, 3, 4, . . .
or $\omega + \omega + \omega$
or $\omega \cdot 3$, etc., etc.

And now we can build up
a HIERARCHY of
transfinite CARDINAL numbers
as follows:

(1) first take the CLASS of
ALL FINITE INTEGERS (positive)
1, 2, 3, 4, . . .
The CARDINAL number of this
WHOLE CLASS of natural numbers
is, as you already know, $\aleph_0$,
which is LARGER than
any of the finite integers
WITHIN the class.

(2) then take the CLASS of
ALL POSSIBLE KINDS of
WELL-ORDERED sets EACH of which has
the power $\aleph_0$,
as stated on p. 173.
The TOTALITY of these is
a DIFFERENT TYPE of set
whose ORDINAL number is
LARGER than that of
any of its CONSTITUENTS
and is represented by Ω.
And the CARDINAL number of
this ENTIRE CLASS
is no longer $\aleph_0$,
but is LARGER than $\aleph_0$,
just as in (1) above
the CARDINAL number, $\aleph_0$, of
the entire class of finite integers
is LARGER than
any of its CONSTITUENTS
since these are all FINITE numbers.
Cantor designated the

CARDINAL number of
this SECOND class by $\aleph_1$.

(3) Similarly the class of
all possible types of
well-ordered sets of power $\aleph_1$
will form a THIRD class
whose power, or CARDINAL number,
Cantor designated by $\aleph_2$,
and so on and so on.

He thus obtained an infinite set of
INCREASING TRANSFINITE
CARDINAL NUMBERS:
$\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_\nu, \ldots$
But even this unlimited sequence of
cardinal numbers
"does not exhaust the conception of
transfinite cardinal number.
We will prove the existence of
a cardinal number which
we denote by $\aleph_\omega$ and which
shows itself to be the next greater
to all the numbers $\aleph_\nu$;
out of it proceeds
in the same way as $\aleph_1$ out of $\aleph_0$
a next greater $\aleph_{\omega+1}$,
and so on,
without end."
This quotation is from one of
Cantor's own papers —
see p. 109 of the book mentioned in
the footnote on p. 78.

It is natural to ask whether
c (which equals $2^{\aleph_0}$, remember?) *

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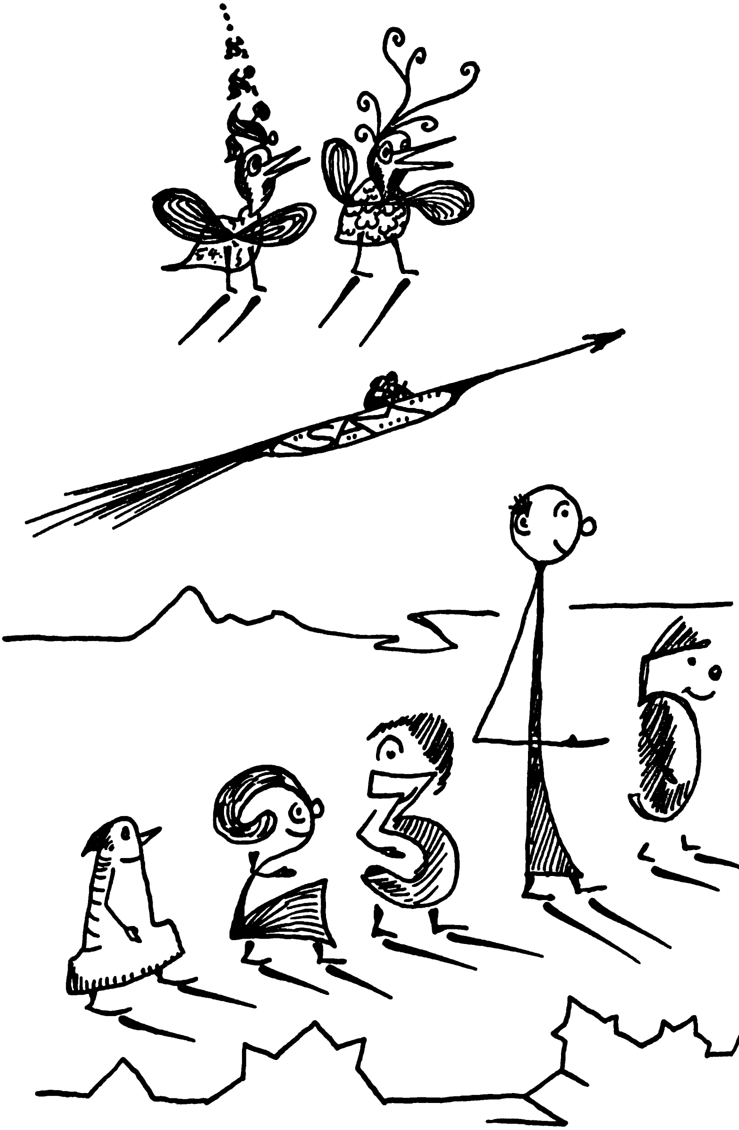
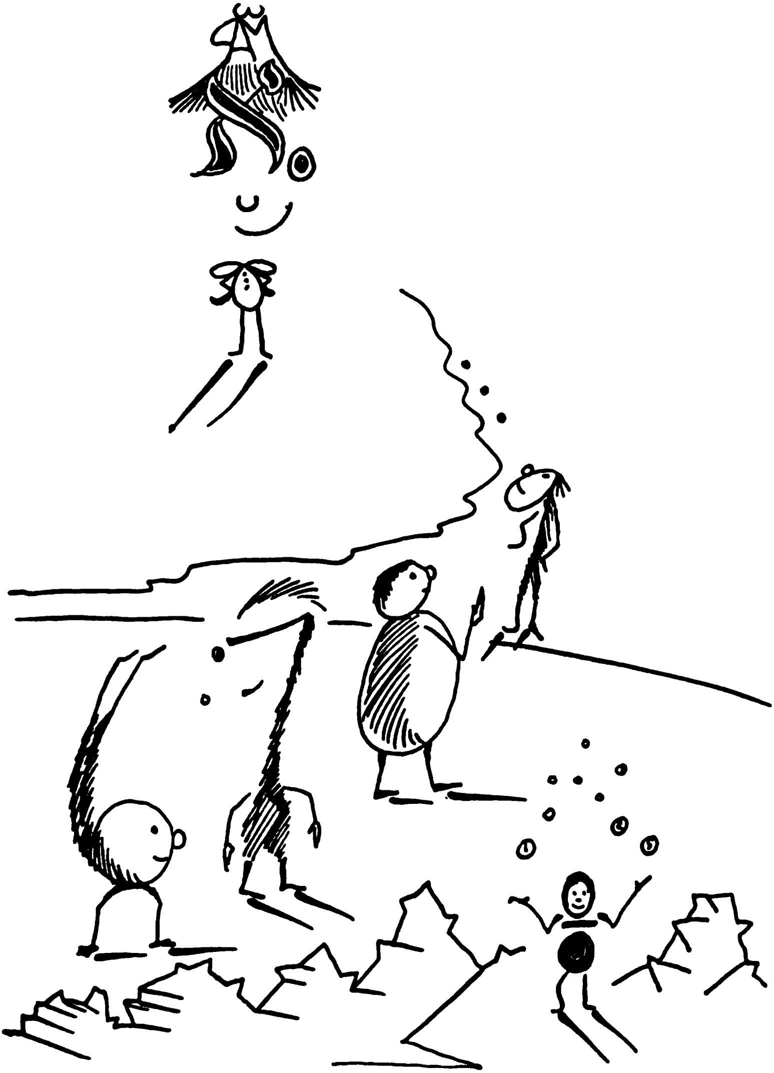
*See p. 133.





is one of this hierarchy of alephs!
Cantor himself ASSUMED that

$$c = \aleph_1,$$

but this question has
NOT YET
been answered to the satisfaction of
all mathematicians!
This is known as
the "continuum problem".
Indeed
"the question whether
every transfinite cardinal number
is necessarily an Aleph-number . . . .
is equivalent to
the question whether
every aggregate (i.e. set)
is capable of being
NORMALLY ORDERED (i.e. WELL-ORDERED)".*

Not only are there still
these outstanding problems
awaiting solution,
but
some mathematicians have even
pointed out
PARADOXES
in Cantor's Theory,
and, as you know,
PARADOXES or contradictions
are INTOLERABLE!
But the Theory has turned out to be
so useful (as you will see)

---

*See E. W. Hobson:
"The Theory of Functions of
a Real Variable", p. 238
(Cambridge Univ. Press).
See also pp. 268-270 in Hobson.

that none but extremists
were willing to
discard the Theory
but tried rather
to put it on a sounder basis
so that at least
the paradoxes which
were pointed out
might be eliminated.

As of now
these efforts
HAVE met with success,
though of course
no one can be sure that
new paradoxes may not
some day arise.
But, for that matter,
this is true even of
ordinary arithmetic —
believe it or not!

These matters will be discussed
in subsequent chapters
of this book,
and constitute
most basic and interesting
considerations,
as you will see.

But, for the present,
in spite of all these
difficulties and doubts,
we MUST NOT
draw the conclusion that
mathematics is unreliable or useless!
The fact is that

it is as reliable as
anything human can be —
which means that
though it is
NOT PERFECT,
still
it leads to so many
VALUABLE RESULTS,
that it would be
little short of madness
to discard it or
even to lose faith in it!
As Einstein has said:
"Alles was wir machen ist falsch."
(Everything we humans do is false.)
And yet you must admit that
by sheer reason —
or rather by sheer SAM
(which includes not only reason, "M",
but also intuition, "A",
and contact with
the physical world, "S",
so that SAM is much more powerful
than reason alone!) —
he was able to give us
the concept of
Atomic Energy,
AND, as you know,
IT REALLY WORKS!
Incidentally
let us not confuse
atomic energy with the atom bomb,
which,
though REALISTIC enough,
is such a horrible
ABUSE of atomic energy,
that we MUST DISCARD this ABUSE of it

WITHOUT, however, discarding
THIS WONDERFUL NEW SOURCE OF ENERGY.
And let us not forget that
it was arrived at by
the very HUMAN SAM,
even though he is
NOT PERFECT!

In short,
let us not be such
PERFECTIONISTS that
we are willing to throw away
a beautiful and useful structure like
Cantor's Theory!
And, on the other hand,
let us not just accept
any old thing,
using as an excuse:
"Nothing is perfect anyway!"

And so
let us realize that
TO DO OUR BEST IS WORTH WHILE,
and
ANYTHING LESS THAN OUR BEST
IS NOT GOOD ENOUGH!

14 A Brief Summary

So far then you realize that:

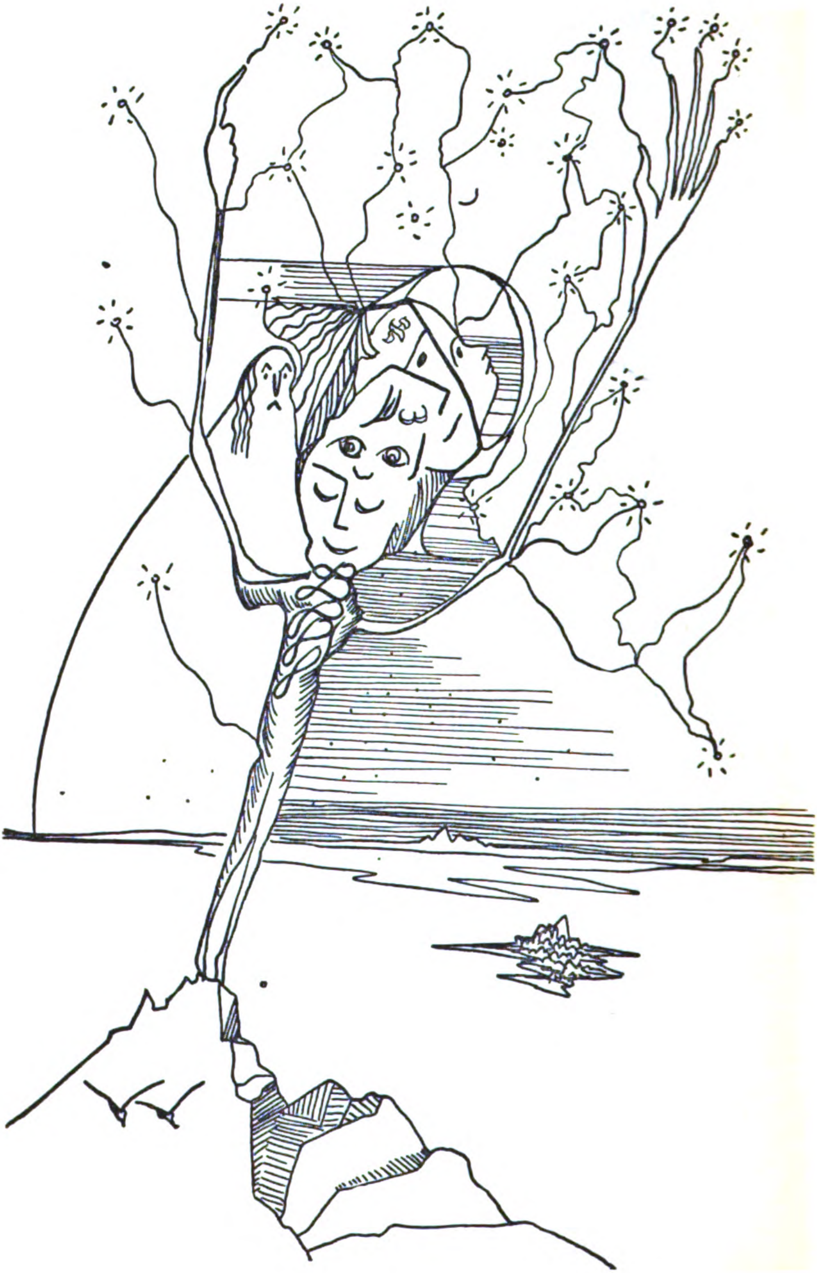
(1) Man's yearning for
the INFINITE
has NOT been fulfilled in the
PHYSICAL world.
Even the
ENTIRE PHYSICAL UNIVERSE
is NOT INFINITE,
so far as we know.
Even the TOTAL number of electrons
in the entire physical universe
is NOT INFINITE.
The old-fashioned idea that
the earth was flat
and extends to infinity —
unless it were a finite disc
with a BOUNDARY, like a fence,
all around it —
turned out to be false,
for as we now know
the earth is a SPHERE
and thus is NOT INFINITE,
and neither does it have
a boundary around it
or, as we say,
it is UNBOUNDED but FINITE.
And similarly
our three-dimensional universe
has also turned out to be
UNBOUNDED but FINITE.
Thus, wherever we look in the

PHYSICAL universe
we have NOT found INFINITY
(see Ch. 2).

Need I add that
those who think they can satisfy
their yearning for INFINITY
in a LIMITLESS GREED for
PHYSICAL possessions or for
PHYSICAL power
are also doomed to
FAILURE,
as in the cases above-mentioned.

(2) Man's yearning for
the INFINITE,
even though it has NOT been
fulfilled in the PHYSICAL world,
has not, however, been abandoned.
Far from it!
But it is still being searched for
in the
MIND and SPIRIT of MAN
(in the A and M of SAM
rather than in S);
and, strangely enough,
it turns out that
this search into the
ABSTRACT
helps man even in the
PHYSICAL domain,
as you will soon see!

As you have already seen,
mathematicians first started with
the idea of a
"potential" infinity, ∞,



which is
APPROACHED but NEVER REACHED.
But even this idea of
a potential infinity
already enabled the mathematician
to keep far ahead of
the "practical" man,
by furnishing the latter with
finite numbers which are
as LARGE or as SMALL as
he would ever require,
by furnishing him with curves
like the "conic sections"
(see Ch. 4),
some of which go off to
infinity,
and yet are so definite in shape
(not at all a mere vague "yearning")
that they can be represented by
precise algebraic equations
from which all kinds of properties
of these curves
may be studied
and applied to all sorts of
practical uses,
as, for example, to
ballistics,
as well as many, many others.
Thus,
the mathematician,
by getting away from
the "practical" man,
and taking an excursion into the
abstract,
even daring to think of
"potential" INFINITY,
then returns to

his "practical" colleague,
laden with new treasures
which turn out to have
many NEW practical uses.
How long will it be before
we begin to understand
how very valuable are the rare souls
who make these excursions into the
ABSTRACT,
and stop calling them
"impractical long-hairs"?
Such ignorance is
APPALLING,
and we should all take a hand
in helping to correct this situation
if we wish to become really
civilized.
And, in this connection,
let us not forget
what a further consideration of
"potential" infinity
did for the study of parallel lines
and for the various changes in
Euclid's famous "parallel postulate",
each of these changes
leading to
NEW, NON-EUCLIDEAN, GEOMETRIES
one of which turned out to be
such a necessity in the
MODERN study of
the PHYSICAL world.*

(3) Man's yearning for
the INFINITE
has, as you now know,

---

*See "The Einstein Theory of Relativity"
by Lillian R. Lieber.

been developed STILL FURTHER
by mathematicians who even
dared to go BEYOND the idea of a
"potential" infinity,
and to consider
"actual" infinity,
meaning
CLASSES or SETS or AGGREGATES
each containing
an INFINITE "number" of members,
thus arriving at
a whole hierarchy of
TRANSFINITE NUMBERS,
both ORDINALS and CARDINALS!

To summarize briefly how
Cantor built up this
marvellous structure of
various INFINITIES,
let me remind you of
several important basic concepts:

(1) Every "actually" infinite set
can be put into
1–1 correspondence with
SOME PART of itself (see pp. 85 and 110).

(2) A "simply-ordered" set (p. 169)
has the following 3 properties:
(a) if a and b are
distinct elements of the set,
then either a precedes b
or b precedes a,
for the elements are arranged in
some DEFINITE order.
(b) if a precedes b in the set,
then a and b are distinct.

(c) if a precedes b and
b precedes c,
then a precedes c in the set.

(3) A "WELL-ordered" set (p. 169)
has, in addition to
the above-mentioned properties
(a), (b), (c),
also the following three properties:
(d) it has a FIRST element;
(e) every element EXCEPT the LAST,
IF THERE IS A LAST,
has an immediate successor;
(f) every fundamental segment of the set
has a limit.

It is important to remember that
a WELL-ordered set has
a definite ORDINAL number.

Let me now recall
a few illustrations of various
actually infinite sets:

I. Take first
the smallest actually infinite set,
that of the natural numbers,
1, 2, 3, 4, ... :
a. its "power" or
CARDINAL number is $\aleph_0$;
b. it is a SIMPLY-ordered set of
order-type ω (p. 165);
c. it is also
a WELL-ordered set*
and its ORDINAL number is ω.

---

*Note that it has properties (d), (e),
but that (f) may be said to be

II. Consider now the set of
all negative integers arranged thus:
. . . . , −4, −3, −2, −1:
a. its CARDINAL number is $\aleph_0$;
b. it is a SIMPLY-ordered set of
type $^*\omega$ (p. 165);
c. it is NOT a WELL-ordered set
(it has no first element)
and therefore has
NO ORDINAL number.

III. Take the set of ALL integers,
positive, negative, and zero,
arranged in order of
increasing magnitude:
a. to find its CARDINAL number,
it is of course permissible to
RE-ARRANGE the order (p. 116) thus:
0, 1, −1, 2, −2, ... ,
and hence its CARDINAL number
is again $\aleph_0$;
b. when left in the order of
increasing magnitude:
. . . . , −4, −3, −2, −1, 0, 1, 2, 3, 4, ...
it is seen to be
a SIMPLY-ordered set of type π (p. 166)
BUT
c. it is NOT a WELL-ordered set
and therefore has
NO ORDINAL number.

IV. Now consider
the set of positive rational numbers:
a. its CARDINAL number is also $\aleph_0$ (p. 94);

---

satisfied "vacuously", since
this particular set does not have
a fundamental segment and therefore
(f) makes no demand here!

b. if arranged in the usual order of
increasing magnitude,
it is a SIMPLY-ordered set of
type η (p. 171)
but NOT a WELL-ordered set;
when so arranged
it is a DENSE set (p. 91)
but NOT a CONTINUUM (p. 104);

c. It can however be so arranged as to
satisfy the conditions of
a WELL-ordered set (p. 169),
and then it is "similar" to
the set of natural numbers,
and therefore also has
the ORDINAL number ω.
Everything that has just been said
about the set of
positive rational numbers
applies also to the set of
rational numbers between
any two given numbers,
say a and b;
also to the set of
ALL POSITIVE AND NEGATIVE
RATIONAL NUMBERS
WITH ZERO INCLUDED,
for,
when the positive ones
are arranged
as on p. 94
each positive one can be
immediately followed by
the corresponding negative one,
so that this totality is also
"countable"
and therefore has
the CARDINAL number $\aleph_0$.

V. Take next
the set of ALL REAL NUMBERS in
the interval from 0 to 1, inclusive.
a. its CARDINAL number is c (p. 125);
b. if arranged in the usual order of
increasing magnitude,
it is a SIMPLY-ordered set of
type θ (p. 171);
c. it has NEVER been shown to be
a WELL-ordered set,
although Cantor BELIEVED that
EVERY set can be well-ordered.
This question is still
under consideration.

VI. Consider next
the various sets whose
ORDINAL numbers are given on p. 173:
a. note that EACH of them is
a "countable" set and therefore
has the CARDINAL number $\aleph_0$;
b. each of them is of course
SIMPLY-ordered (p. 169);
c. indeed each of them is also
WELL-ordered (p. 169),
having the ORDINAL numbers
given on p. 173.

VII. The TOTALITY of ALL sets each
having the cardinal number $\aleph_0$
(i.e. ALL those in VI. above)
forms a NEW SET:
a. its CARDINAL number is $\aleph_1$ (p. 175)
which is GREATER than $\aleph_0$;
Cantor ASSUMED that $\aleph_1 = c$
but this a question which is
still under consideration (p. 178);

b. it is a SIMPLY-ordered set;
c. it is also WELL-ordered and
its ORDINAL number is Ω (p. 174).

VIII. Similarly the CLASS of
ALL possible types of
WELL-ordered sets having
the cardinal number $\aleph_1$,
form a THIRD class,
having the CARDINAL number $\aleph_2$,
and so on and so on,
thus obtaining an infinite set of
INCREASING TRANSFINITE CARDINAL
NUMBERS:

$\aleph_0, \aleph_1, \aleph_2, \ldots$ (See p. 175).

You saw* how
Cantor "operated" with
infinite sets
leading to some amazing results!

Incidentally
you have also seen
how valuable is our
common number system (p. 127),
and that there are also
other valuable systems based on
a radix other than 10 (p. 128).
You have also become acquainted
not only with
the FOURTH dimension,
but also with HIGHER dimensions, —
and even with the concept of
an INFINITY of dimensions (p. 161)
and the practical value of these (p. 157).

---

*Ch. 9, Ch. 11, Ch. 13.

But before going any further into the
Cantor Theory of Transfinites
or "actual" infinities,
let us stop for a moment
to consider one phase of its
"legitimacy"
(we shall consider later
other phases of
this important question).

15 Is It Legitimate?

As you have seen,
there are still
some outstanding questions
in the Cantor Theory
which have not yet been answered
(see pp. 178, 191).
But
what about the rest of the theory?
What about the very concept of
an "actual" set consisting of
an infinite "number" of members?
Has this basic idea been
adversely criticized?
by whom?
Have any of the criticisms been
successfully refuted?
Is the idea practical and useful?

Let us see.

In the first place
it is important to realize that
the question as to the permissibility
of the concept of
"actual" infinity
as distinguished from
"potential" infinity
(see Ch. 2, etc., etc.)
is a very old question,
which arose long before Cantor:
thus
"In 1831 Gauss expressed his

'horror of the actual infinite'
as follows:
'I protest against the use of
infinite magnitude as
something completed,
which is never permissible in mathematics.
Infinity is merely
a way of speaking . . .'"[*] —
Cantor was not born until 1845.

Further,
Kronecker, a contemporary of Cantor,
said:
"God made the integers,
all the rest is the work of man."[‡]
And again:
"All results of the profoundest
mathematical investigation
must ultimately be expressible
in the simple form of
properties of the integers."[ϕ]
Thus Kronecker declared that
there is no such number as

$$\sqrt{2} \text{ or } \sqrt{-1},$$

that only the positive integers
are permissible,
and that all other "numbers"
must be expressible in terms of
the positive integers
(the "natural" numbers).
This forced his contemporaries,
like Dedekind, Weierstrass, Cantor,

---

[*] "Men of Mathematics" by E. T. Bell p. 556.

[‡] P. 477 ibid.

[ϕ] P. 466 ibid.

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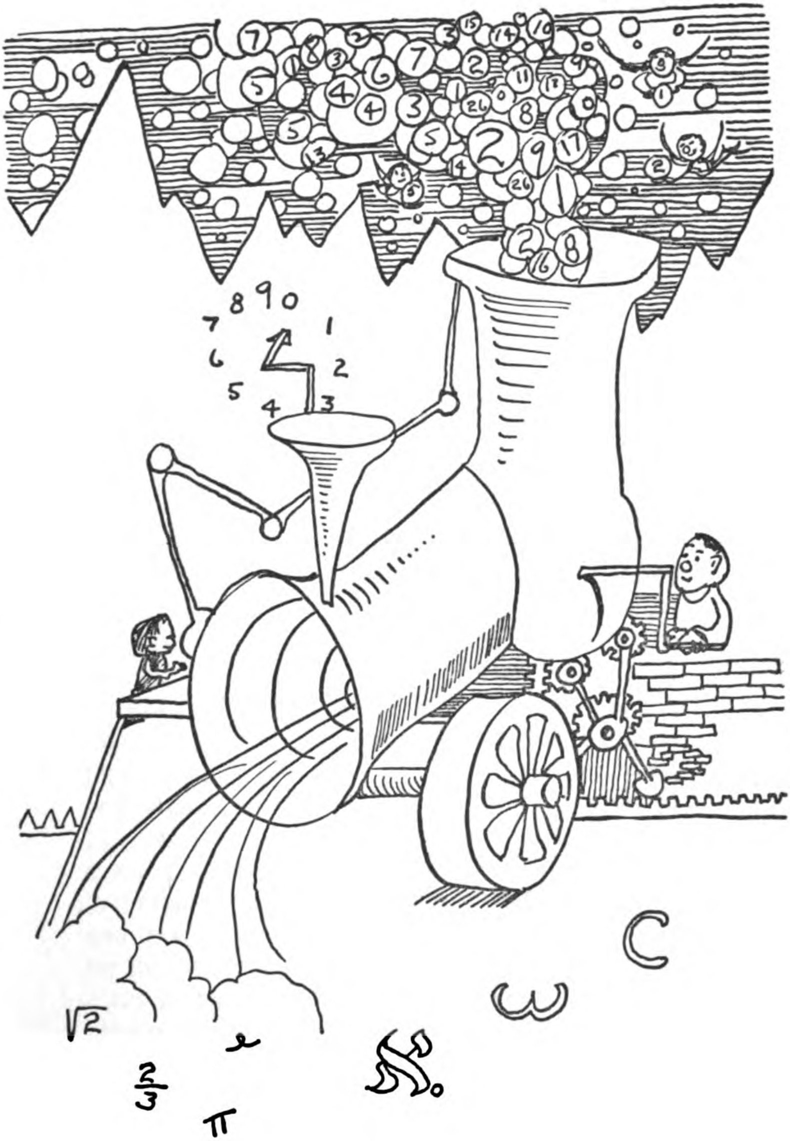
to see how this can be done —
even though they took sharp issue*
with Kronecker's viewpoint on
irrational numbers,
and hence on the whole subject of
"The Theory of Functions"
which is so basic for mathematics,
both "pure" and "applied",
today!

Now you know that
a RATIONAL number is one which
can be expressed as
a RATIO of two INTEGERS,
like 7/11, etc., etc.
Thus the rational numbers
fulfill the demand made by
Kronecker.
And now let us see how
this demand was fulfilled also for
ALL the REAL numbers:
Dedekind did it by means of
the "Dedekind cut",
as follows:

Imagine the RATIONAL numbers,
negative, zero, and positive,
all arranged in
order of magnitude.

---

*Thus Weierstrass wrote in 1885 to
Sonja Kowalewski
(herself a great mathematician):
"But the worst of it is that
Kronecker uses his authority
to proclaim that
ALL those who up to now have
labored to establish
the theory of functions
are sinners before the Lord."
(See p. 480 of Bell's "Men of Mathematics".)


8 9 0
7 1
6 2
5 4 3
√2
e
2/3
π
ℵ₀
ω
c

Now "cut" this set into
two parts, R_1 and R_2,
such that all the numbers of R_1
are less than all those of R_2.
If this "cut" hits one of the numbers,
say the number 1/2,
"right on the nose",
this number may be considered to be
the LAST one of R_1 or
the FIRST one of R_2 —
in either case it is obviously
a RATIONAL number.
But suppose the "cut"
does NOT hit one of the
original set of rational numbers
"right on the nose",
then obviously
R_1 HAS NO LAST number,
and R_2 HAS NO FIRST number:
Such a "cut" then
DEFINES a REAL number which is
NOT RATIONAL
and is therefore called "IRRATIONAL".
Note that this definition of
REAL number,
by means of a "cut",
makes it depend upon
the RATIONAL numbers,
since this is the set we started with,
and these in turn depend upon
the INTEGERS (p. 196),
so that all these kinds of numbers
are thus made to
stem from the integers.

Now let us see how
Cantor defined the REAL numbers

in terms of the INTEGERS:
Consider the sequence of
RATIONAL numbers in the interval
from 0 to 1
including the 0 but not the 1:

$$x_1, x_2, x_3, x_4, \ldots\ldots, x_n, x_{n+1}, \ldots\ldots$$

where $x_1 = 1/2$ and is thus
half-way between 0 and 1,
$x_2 = 3/4$ and is
half-way between x_1 and 1,
$x_3 = 7/8$, half-way between x_2 and 1,
and so on and so on.
It is obvious that
all subsequent x's will be
nearer and nearer to 1,
but will never go beyond 1
and will not even reach it.
Now, if you choose a positive number,
call it ϵ,
then
no matter how small ϵ is,
and you may choose it
AS SMALL AS YOU PLEASE,
there is sure to be an n so large
that $(1 - x_n) < \epsilon$
and of course then
$(1 - x_{n+1}) < \epsilon$,
since x_{n+1} comes after x_n and
is therefore nearer to 1 than is x_n;
and this is also the case
for the difference between
1 and all subsequent x's.
Hence we call 1 the "limit" of
the set of rational numbers given above.
Similarly
EVERY real number

is the LIMIT of
SOME sequence of RATIONAL numbers.
This is Cantor's definition of a
REAL number,
and is equivalent to
the "Dedekind cut" definition
given on p. 196,
since both lead to
the SAME set of numbers,
namely, the "REAL" numbers.*
So you see that both
Dedekind and Cantor
made all the REAL numbers
(both rational and irrational)
EXPRESSIBLE in terms of the INTEGERS;
but in order to accomplish this
they used the concept of
"actual" infinity
when they started with
an infinite set of rational numbers
as described on pp. 196 ff.
Thus they did NOT REJECT
"actual" infinity,
as did Gauss much earlier,
in 1831 (see p. 194)!

And so we see that
the rejection of "actual" infinity
would mean the rejection of
irrational numbers —
which implies that
it is impossible to
solve an equation like $x^2 - 2 = 0$
(since its "roots" are $\sqrt{2}$ and $-\sqrt{2}$,
both irrational numbers)!
And of course it follows that

---

*See E. W. Hobson: Theory of Functions of a Real Variable,

complex numbers like $x + iy$ —
where x and y are ANY real numbers,
including of course irrational ones —
would also be ruled out,
and hence it would be impossible also
to solve an equation like

$$x^3 = 1$$

whose roots are
$1, \quad (-1 + \sqrt{-3})/2, \quad (-1 - \sqrt{-3})/2,$
the two latter being complex numbers
which would be ruled out!
Thus we would be obliged
to throw out
the beautiful and general theorem that
EVERY ALGEBRAIC EQUATION
of the nth degree
has exactly n roots —
a theorem which is so
very important and useful
not only in pure mathematics
but also in the sciences in which
mathematics is applied to
the solution of equations
as well as in many other ways.

And so we see that
the rejection of "actual" infinity
would destroy an enormous amount of
standard and useful mathematics!
Now of course this does not
make the concept of "actual" infinity
necessarily acceptable,
but certainly
mathematicians would think twice,
and then some,
before they would lightly throw away

a substantial part of
their beautiful and useful structure,
but would prefer
to examine the concept of
"actual" infinity,
as well as other basic concepts
THOROUGHLY
and,
if they find any flaws in them,
to eliminate the flaws,
as far as is humanly possible.

You will soon see that flaws in
Cantor's Theory of Sets
(or "Mengenlehre", as it is called
in German)
HAVE been found and
HAVE been eliminated —
though of course
there is no guarantee that
no new flaws will ever be found —
but, for that matter,
that is true of
all human inventions,
including EVEN
scientific and mathematical domains.
And so
mathematicians are
PREPARED in their minds for
the possibility,
and indeed for
the PROBABILITY that
CHANGES WILL BE NEEDED IN THE FUTURE.
And, in the meantime,
THEY DO THE BEST THEY CAN,
and find that this "BEST" is
amazingly beautiful and useful.

But before going into
the above-mentioned flaws and
the remedy for their elimination
from Cantor's Mengenlehre,
let us look at
some very interesting properties of
some very interesting sets.

16 Some Very Interesting Infinite Sets

Now
a set may or may not
contain ALL its "limit" points:
thus the sequence of
rational numbers on p. 199
has only one limit point,
namely, the number 1,
and this is
NOT contained in the set,
as the set was chosen on p. 199.
Similarly,
the set of ALL rational numbers
contains an infinity of its limit points,
since EVERY real number,
and hence every rational number,
is the limit of
SOME sequence of rational numbers (p. 200).
But this set (the rational numbers)
does NOT contain
ALL its limit points,
since, for example,
it does not contain $\sqrt{2}$
although this, being a real number,
is the limit of
SOME set of rational numbers,
as is true of
EVERY real number (p. 200).
When a set DOES contain
ALL its limit points,
it is called a "closed" set —

otherwise, it is an "open" set.

In general,
if a set, H,
is part of a set, G,
and is such that
every point of G is
a limit point of H,
the set H is said to be
"dense in G".
Thus,
the set of rational numbers
is dense in
the set of real numbers.
But if H is identical with G,
and if every point of G is
a limit point of G,
then G is said to be
"dense in itself".
Note that this
does NOT necessarily mean
that the set contains
ALL its limit points.
Thus the set of rational numbers
is dense in itself.
This concept is consistent with
the idea that
in a dense set,
between any two of its elements,
there is an unlimited number of
other elements of the set (p. 90).

Now if a set is BOTH
closed (i.e. contains ALL its limit points)
AND dense in itself,
it is said to be
a PERFECT set.

Thus, for example,
the set of REAL numbers
is a PERFECT set.
Whereas
the set of RATIONAL numbers
is DENSE in itself but
NOT PERFECT.
And
the set mentioned on p. 199,
since it has only
one limit point,
namely, the number 1,
which is NOT included in the set,
is therefore
NEITHER PERFECT NOR DENSE.
While this same set
BUT WITH THE 1 INCLUDED
is PERFECT but NOT DENSE.

Some of these properties of sets
will be helpful later in
discussing some interesting "paradoxes"
which have come down to us
from antiquity,
and are still being
ineffectively fought over by
many a young student and
old would-be philosopher!
You will find that
the theory of sets,
as well as other branches of
Modern mathematics
will help you
in considering those old problems
as well as some new ones.

Now consider this ASTONISHING set,

given by E. V. Huntington:*
the set of rational numbers between
0 and 1,
arranged in ascending
order of magnitude, and
all of them colored BLUE;
this, as you already know (p. 205),
is a dense set.
Now, take the set of
rational numbers between 0 and 1
again arranged in
ascending order of magnitude, and
all of them colored RED;
this is of course also a dense set.
Finally,
put these two sets together
in such a way that each BLUE number is
IMMEDIATELY followed by
the corresponding RED number:
thus the BLUE 1/2 is
immediately followed by
the RED 1/2,
etc., etc.
Now wouldn't you think that
if you "pack in" a dense set
in amongst another dense set,
the resulting set would be
even more dense?
And yet,
since each BLUE number has
an IMMEDIATE successor,
namely,
the corresponding RED number,
so that between these two
you cannot "pack in"
even a single other rational number,

---

*Annals of Math. (2) vol. VII. (1905), p. 15.

RED or BLUE,
so that the resulting set is
NOT DENSE at all!
And, of course,
it is also NOT PERFECT,
since it does NOT contain
ALL the REAL numbers,
and hence does NOT contain
All its limit points.

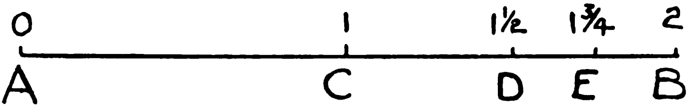
You must admit that
Huntington's RED-BLUE set
is an ingenious invention,
since
the BLUE set alone is already
DENSE to begin with,
and then when you "pack in"
the RED set,
which is itself also a DENSE set,
what happens? —
the set LOSES its denseness!
Now surely
you would not have arrived
at this conclusion by means of
"common sense" alone,
would you?

And so you have seen
various kinds of interesting sets
having the properties of being
dense or perfect or both or neither, etc.

Let us next see how this study of sets
can help to clarify a problem which
has come down to us from antiquity,
and which has troubled many a restless soul
through the ages.

17 Applications: Zeno et al.

Suppose the distance from A to B
is 2 feet:



C is half-way between A and B,
D is half-way between C and B,
E is half-way between D and B,
and so on, —
continue to approach
closer and closer to B
by cutting in half
the interval remaining at each step.
Now suppose that
a point is at A in the beginning,
at C after one second,
at D after another 1/2 second,
at E after another 1/4 second,
etc., etc.,
approaching B
in the way described above,
the time for each step being
half of that for the previous step.
Now the question is:
(a) Will the point ever reach B?
(b) If so, when?
So far as question (a) is concerned,
note that

the points A, C, D, E, ...
constitute an infinite discrete set,
having only one limit point,
namely, point B (see p. 209).
IF point B is INCLUDED in the set,
the set is a PERFECT set (p. 206);
IF B is NOT included in the set,
the set is not a perfect set (p. 206).
In the former case,
B would be one of the positions
assigned to the moving point;
in the latter case,
the moving point would NOT
occupy the position B,
since B is then not included
in the set of positions.
Thus if the above set of points is
CHOSEN as a perfect set,
A, C, D, E, ... , B,
it would have the ordinal number $\omega + 1$ (p. 167);
whereas
if the set of points is
CHOSEN without its limit point B,
thus: A, C, D, E, ...
it would have the ordinal number ω (p. 164).
In short,
the question (a) on p. 209 is
NOT a LEGITIMATE question
with the inadequate data given on p. 209.
But
If we include in the data
the stipulation that
the set of positions to be occupied by
the moving point
is to be
the PERFECT set described above
(i.e., including the limit point, B),

then the answer to question (a)
would be "Yes".
In that case,
we could proceed to find
the answer to (b) on p. 209 thus:
$1 + 1/2 + 1/2^2 + 1/2^3 + \cdots$
Since this is an
infinite geometric series,*
its sum may be found by the formula

$$S = a/(1 - r)$$

where a is the first term of the series
(here $a = 1$)
and r is the common ratio
(here $r = 1/2$).
Hence $S = 1/(1 - 1/2)$
$= 1/(1/2) = 2$ seconds.
If, however,
the set of positions is so defined as
NOT to include its limit point, B,
then the answer to (a) on p. 209 is
"No",
in which case
the answer to (b) is
"Never".

In other words,
questions like the ones on p. 209
must be ASKED properly,
giving all necessary stipulations,
before an answer may be expected!

Take, for example,
a similar problem given by
E. J. Moulton on p. 342 of

*Look this up in any book on
elementary algebra.

the June 1948 issue of
the American Mathematical Monthly
published by
The Mathematical Association of America:
First he proposes:
"Problem 1. I am sitting at a table
four feet wide. I place my knife at
one edge of the table, then half-way
across the table, then half-way from
there to the other edge, then half-way
from there toward that edge, and so on
indefinitely. How long a time is it
before the knife is at that edge? We
understand that we are talking about
a mathematical knife having no width,
that the measurements indicated at the
various steps are mathematically exact,
and that the problem is strictly
mathematical."
He then says that a bright ten-year-old
may realize that since there was nothing
said about time in the data, no
conclusion could be drawn about how
long a time it would take, and he adds:
"Your colleagues may be taken aback by
this remark."

I hope you realize
how important is
the "bright ten-year-old's" comment.

Moulton then proceeds to add to
the data of his Problem 1
"the statement that it took one second
for the knife to be moved in the first
step, a half a second for the second
step, a fourth of a second for the

third step, and so on, with the general
rule that it took $1/(2^{n-1})$ seconds for
the nth step, where n is a positive
integer. NOW what is the answer to the
question of Problem 1?"
He then says:
"Most of your friends, after a little
thought, will say that the proper
answer is two seconds."

Moulton then proceeds to
criticize this answer as follows:

"My answer is that from the data
no conclusion can be reached, as to
when the knife reaches the far edge of
the table. Let s ft. be the distance
of the knife from the edge where it
started, after t seconds. The value of
the dependent variable t is given for
a discrete set of values of the
variable s. The value $s = 4$ is not
included in this discrete set, and hence
we do not know what t is when $s = 4$."

In other words,
As the problem was stated,
the value $s = 4$ was
NOT included in the set!
He then goes on:

"If we add further hypotheses, to the
effect that t is a monotonic increasing
function of s on the range $0 \leqq s \leqq 4$," —

note that here
$s = 4$ is now specifically

INCLUDED in the set.—

"and is continuous at $s = 4$,"

which means that
$s = 4$ is a limit point

"we might conclude that $t = 2$
when $s = 4$"

as on p. 211.
And he ends up by saying:

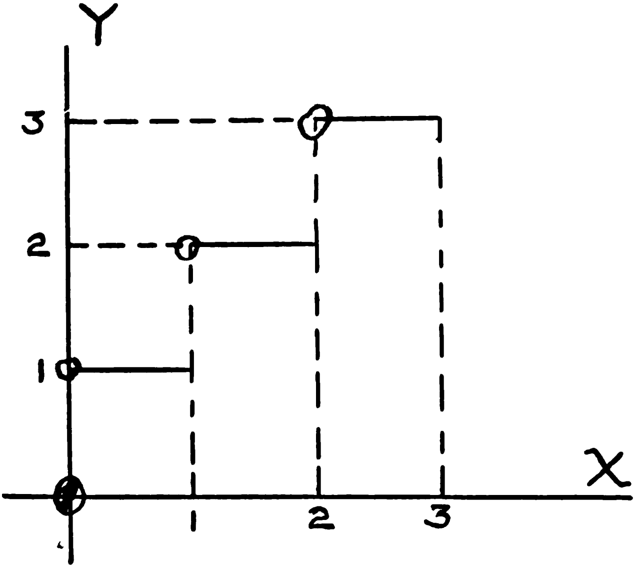
"Otherwise who knows whether t makes
finite jumps!"

Note that this problem,
like the one on p. 209,
must be very delicately stated,
USING THE KNOWLEDGE OF
THE VARIOUS PROPERTIES OF SETS
as stated on previous pages.

Suppose now that we have
y a function of x
with the graph shown on p. 215.

Here, when $x = 1$, y is either 1 or 2,
BUT
one may deliberately DEFINE that
$y = 1$ when $x = 1$,
as, for instance,
if this function represents
the cost ($y = 1$) of mailing
one ounce ($x = 1$) or
fraction thereof,
but as soon as you have

a package OVER one ounce,
the price jumps to y = 2
and stays on that level up to and
including x = 2 ounces, etc.



Or
suppose this same function
represents the salary (y = 1)
a person receives for the first year
from x = 0 to x = 1 but
NOT including x = 1,
for, as soon as x = 1
(namely, the beginning of the new year),
the salary jumps to y = 2 and
stays on that level UP TO x = 2,
then jumps again AT x = 2,
etc.
In other words, in this case

when $x = 1$,
then $y = 2$ instead of $y = 1$ as
in the case where
this function represented
mailing cost as on p. 214.

Thus,
where there is
a discontinuity in y
(as for $x = 1$ on p. 215)
for a given value of x,
one may SELECT one of the y - values
in accordance with
the MEANING of the function in question
when it is applied to
a given problem.

Similarly,
in the case of life insurance,
the cost of the premium depends upon
one's age,
changing from year to year;
and, if the policy is being written
on a certain date,
your age is taken as
the year of your NEAREST birthday
(it may be your last PREVIOUS BIRTHDAY,
or your NEXT BIRTHDAY,
whichever is NEARER)
and,
if the date of the policy
happens to be EXACTLY half-way
between birthdays,
it is the privilege of
the insurance company to
figure it from EITHER birthday.
And if this same function were used

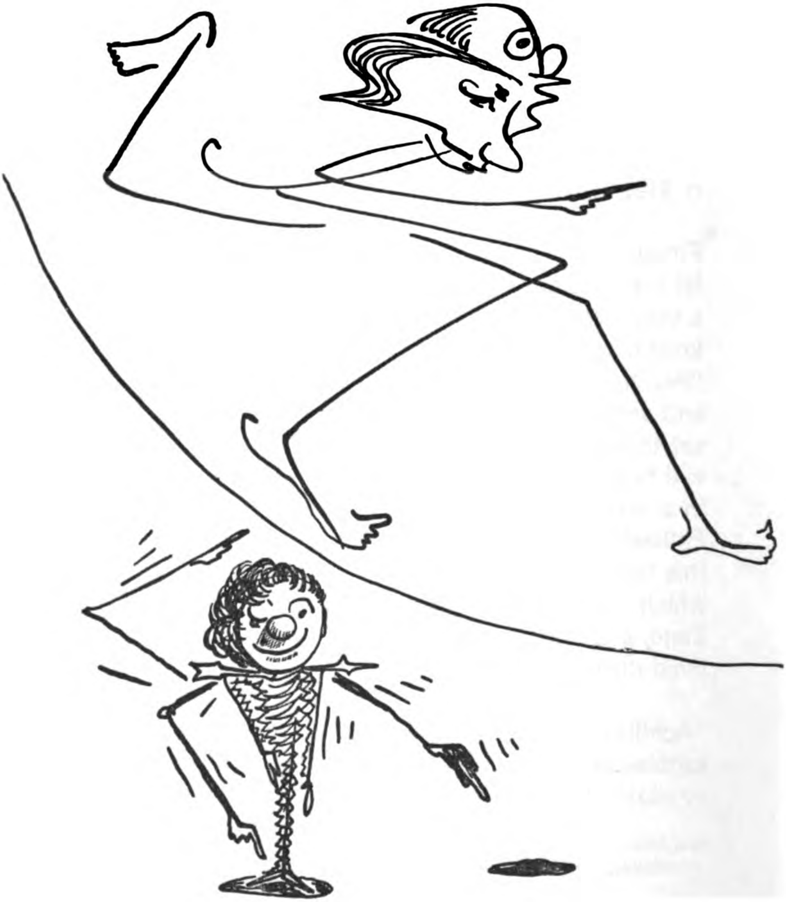
for STILL ANOTHER PURPOSE,
it would be perfectly legitimate,
from the mathematics point of view,
to choose $y = 1\frac{1}{2}$, when $x = 1$,
or $y = 1\frac{1}{3}$, or $y = 3$ or
y equal to ANY other value,
DEPENDING on
the PRACTICAL issue involved!
Only,
one would have to
STATE EXPLICITLY
what value of y is appropriate
wherever there is such a
"discontinuity" or break in the graph
as there is at $x = 1$, $x = 2$, etc. on
p. 215.

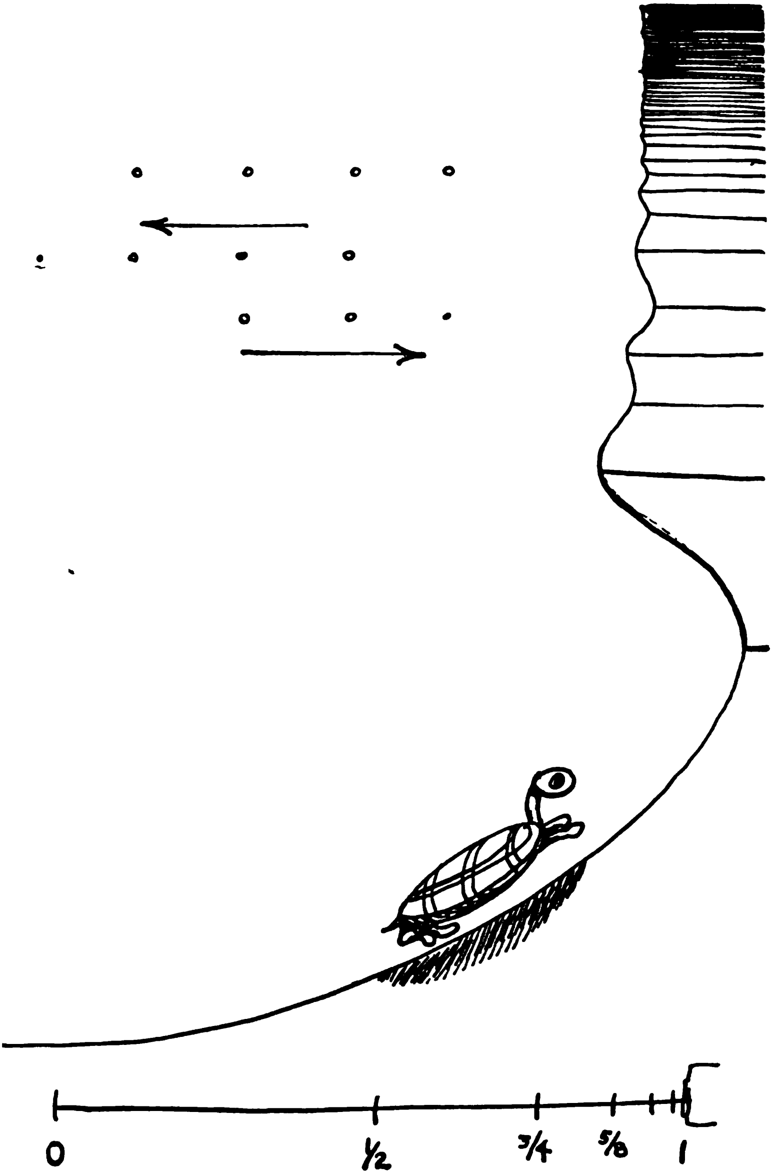
Finally,
let me consider one more problem, —
a very old and famous one, —
known as the problem of
"Achilles and the Tortoise",
and see how a consideration of
set-theory
will help us to think straight
in a very "tricky" situation:
Following is a statement of
this tantalizing problem,*
which was originally proposed by
Zeno, a Greek philosopher, who
lived during the 5th century B.C.:

"Achilles running to overtake a crawling
tortoise ahead of him can never
overtake it, because he must first

---

*See "Men of Mathematics" by E. T. Bell,
4th printing, p. 24.

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1

reach the place from which the tortoise started; when Achilles reaches that place, the tortoise has departed and so is still ahead. Repeating the argument we easily see that the tortoise will always be ahead."

Now we know perfectly well that
there must be something wrong
with this line of argument,
for we know that Achilles,
who was a professional runner,
must surely overtake and pass
the tortoise —
just as an automobile
surely can and does
overtake and pass
another car ahead of it
if the first one travels faster
than the car in front,
and yet the above reasoning
in the Achilles problem
would argue that
the slower car would
always remain ahead!

Now what is wrong with
the argument?
Well, if you consider it carefully,
you will see that
the difficulty is similar to that in
the problem on p. 209 and
in Moulton's problem (pp. 211 ff.),
namely:
a set of positions is selected by Zeno
in such a way that
they approach a limit which is

NOT included in the set —
in other words,
he deliberately selects
a set of positions
whose "limit" is the very point
at which
Achilles actually does overtake
the tortoise,
and,
by deliberately omitting this point
from the set of positions,
he is forced to conclude that
Achilles can never overtake
the tortoise.
But let us look at the problem
in the following way instead:

Suppose, for instance, that
A travels at a uniform rate which is
10 times the rate of T;
then the distance traveled by A is
10 times the distance traveled by T in
a given time, thus:

$$D_A = 10d_T.$$

And suppose further that
at the beginning
they were 18 feet apart;
then

$$10\,d_T = 18 + d_T \qquad (1)$$

as seen from the diagram on p. 222.

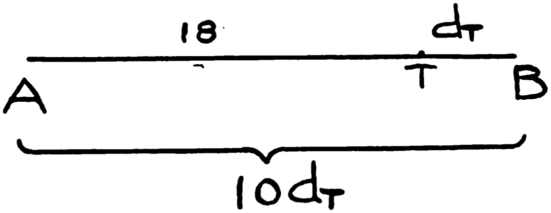
Hence, from (1),

$$9d_T = 18$$

and

$$d_T = 2.$$

In other words,
after the tortoise (T)
has gone a distance of 2 feet,
Achilles (A) will have gone 20 feet
and will just overtake T at B
as any child can see.
Now of course
this way of looking at the problem
is most elementary
and Zeno certainly would
NOT have denied the correctness of it.



BUT
what he proposed in his problem
was a challenge to
FIND THE ERROR in his REASONING which
SEEMS so correct!
And,
if you look at it from the viewpoint of
the theory of sets,
you can see that he selected only
certain positions
which constitute
an infinite set of points having
a limit point which
was NOT included in the set.
Thus he selected positions of
the tortoise which were all
LESS than 2 feet,

he selected positions of Achilles
which were all LESS than 20 feet,
and therefore selected time-intervals
all LESS than the time required for
Achilles to overtake the tortoise!
Compare this with the answer
"Never" on p. 211.

Perhaps the following will
clarify the matter further:
Suppose the following table represents
the distances (d) traversed by an object
in certain times (t):

d	t
1	1
$1\frac{1}{2}$	$1\frac{1}{2}$
$1\frac{3}{4}$	$1\frac{3}{4}$
$1\frac{7}{8}$	$1\frac{7}{8}$
:	:
:	:

Thus in one second, the distance is 1 ft.
in $1\frac{1}{2}$ seconds, the distance is $1\frac{1}{2}$ ft.,
in $1\frac{3}{4}$ seconds, the distance is $1\frac{3}{4}$ ft.,
etc., etc.,
each time adding on
half the time left up to 2 seconds,
and half the distance left up to 2 ft.
Now here we have
two discrete sets:
the set of d's and the set of t's —
each is approaching the limit 2,
and
if we continue writing
the numbers in the two columns
in the way described above,

we shall of course
NEVER reach 2 in either column,
even if we keep on
writing the numbers
till doomsday,
BUT
if we ASSUME that
the relationship between d and t
may be expressed by

$$d = rt$$

where r is a constant (here $r = 1$),
and that d is
a monotonic increasing function of t
on the range $0 \leqq t \leqq 2$
and is continuous at $t = 2$
(that is to say,
2 is a limit point of the set of
t-values on p. 223,
and 2 is INCLUDED in the set
by the assumption of
the range specified),
then
we COULD conclude that
$d = 2$ when $t = 2$.
Now
these assumptions MUST be made
in considering a problem in
uniform "motion" —
as in the Achilles problem —
BECAUSE
otherwise
we reach what we know to be
the "unrealistic" conclusion that
Achilles cannot overtake the tortoise.
In other words,
we must never use an "M" (reasoning)

which contradicts the "S" (observations)
of the problem in question —
because such an "M" is simply
NOT adequate for the problem.
Thus, by introducing
the further assumptions
mentioned above,
the contradiction CAN,
in this case,
be eliminated.
And so,
we must tackle a problem
not with "M" alone,
or "A" (intuition) alone,
but need a BALANCED SAM
in order to avoid contradictions.

Thus we see that,
as usual,
the mathematician is
a step ahead of
the "practical" man:
the mathematician has elaborated
number systems by means of which
it is possible to write
a number which is
AS LARGE AS YOU PLEASE (p. 28),
or
AS SMALL AS YOU PLEASE (p. 28),
and he then introduces
his "practical" friend to
this sumptuous "smorgasbord",
inviting him to select a plate and
help himself to whatever he needs!

And similarly,
the mathematician has prepared

and served up
a sumptuous "smorgasbord" of
infinite sets,
and again invites his
"practical" colleagues to
help themselves to
whatever they require —
but of course they must
choose appropriately
if they are to be successful!
Thus if the "practical" man
ignores the practical data ("S"),
and chooses a line of reasoning ("M")
which ignores the practical data,
how can he expect to get
the right "answers"?
And why, pray,
should he then blame it on
the innocent mathematician,
when it is the "practical" man himself
who changes his own SAM to sAM or AM,
thus turning himself into
a pathological schizophrenic
by "splitting" his personality?! —
as in the problem of
Achilles and the tortoise
discussed on pp. 217 ff.

And now let us look further into
the subject of
"discontinuous" functions (p. 217)
where you will find more
amazing sets
so wonderfully treated
that it is now possible to
integrate functions which
cannot be integrated by means of

Newton's Integral Calculus,
beautiful and useful as that is.
And this Modern integration is,
as usual in mathematics,
a thing of BEAUTY
as well as of
the utmost practical importance.

Let us see.

But before going into
the modern integration of
discontinuous functions,
let us take a brief look at
the Calculus of Newton and Leibnitz*,
both the Differential and Integral
Calculus,
of which one should have
some understanding
in order to better appreciate the
modern integration of
discontinuous functions.

---

*It is interesting to note that
this Calculus was created by both
Newton (an Englishman) and
Leibnitz (a German)
at about the same time (17th century)
but quite independently of each other!
Strangely enough this kind of coincidence
has happened more than once in the
history of mathematics —
as, for example, in the case of
Hyperbolic Geometry which was created by
Lobachevsky (a Russian),
Bolyai (a Hungarian), and Gauss (a German)
all independently of each other (p. 64).

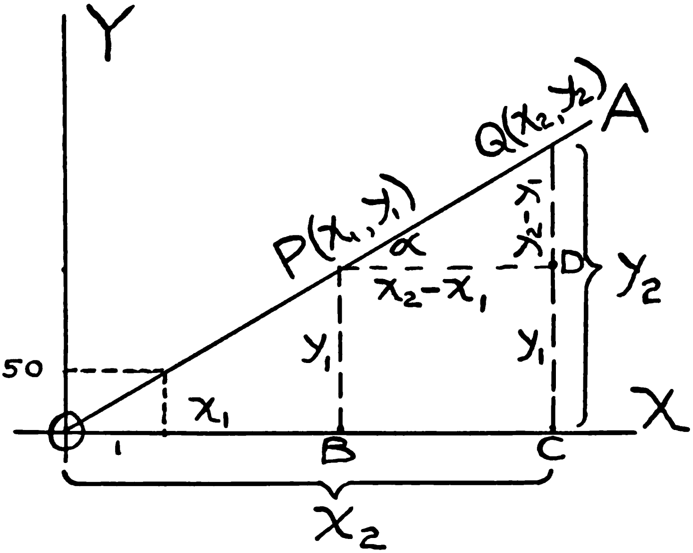
18 The Calculus of Newton and Leibnitz

Consider now the function

$$y = 50x$$

in which $0 \leqq x < \infty$,
that is,
x has a range which includes
all positive real numbers and zero,
and
for each value of x
there is a definite value of y,
namely, 50 times as great as
the value of the x.
In such a case
we say that
y is a continuous function of x.
This particular function
may be represented by
the graph on p. 229,
where the line OA
(prolonged as far as you please)
is a STRAIGHT line.
Now consider ANY two points, P and Q,
on this line,
and draw PB and QC, both perpendicular to OX,
and PD perpendicular to QC.
The coordinates of P and Q are
(x_1, y_1) and (x_2, y_2), respectively,
(as is explained in any book on
elementary algebra).
Now, as you know from

the very first few
elementary lessons in trigonometry:
in the right triangle PDQ
the ratio $(y_2 - y_1)/(x_2 - x_1)$
is called the tangent of angle α,



and we express this definition
more briefly by

$$\Delta y/\Delta x = \tan \alpha^*,$$

Here Δy means the DIFFERENCE
between the y-values of
the points Q and P,

---

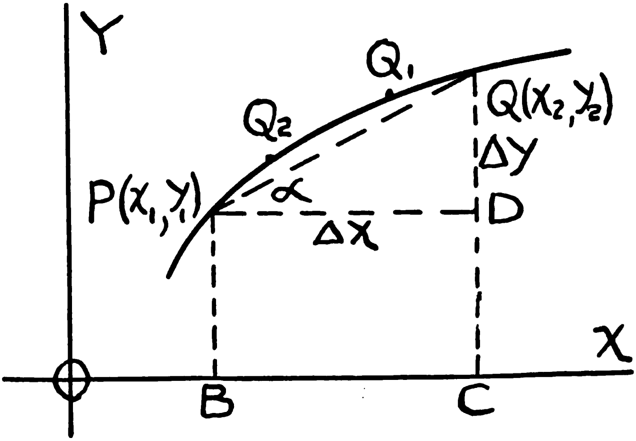
*Δ is pronounced "delta",
as you doubtless know if you,
or your friends,
are members of any
Greek letter fraternity or sorority,
and is the Greek letter for our D
and is used to represent
the word
"DIFFERENCE".

that is, $\Delta y = y_2 - y_1$;
and similarly $\Delta x = x_2 - x_1$,
and of course
"tan" is short for "tangent".
Since in this particular case
the graph, OA,
happens to be a straight line,
the angle α would be the same
no matter where you choose
P and Q on OA.

But now suppose that

$$y = f(x)$$

(read "y is some function of x")
which is continuous and
such that its graph is
NOT a straight line,
and we still have $0 \leqq x < \infty$ and
a definite value of y for
each value of x,
thus:



Here PDQ is no longer a triangle,
since arc PQ is not a straight line.
If, however, you now join P and Q by
a straight line forming angle α with
the line PD,
then we have, as before,

$$\Delta y/\Delta x = \tan \alpha.$$

BUT
if you let Q approach P,
as at Q_1,
and draw again
the dotted auxiliary lines
from this new position,
you would of course have:

(1) C_1 nearer to B than C was,
(2) D_1 nearer to P than D was,
(3) in the triangle PD_1Q_1
the angle α_1 would be
different from α,
(4) and the New $\Delta y/\Delta x = \tan \alpha_1$
would be different from
the original $\tan \alpha$.

This is all very clear, is it not?

And if you came still nearer to P,
say at Q_2,
these four remarks would still hold,
would they not?

Now,
the lines PQ, PQ_1, PQ_2, etc., etc.,
may be called "chords" since they
join two points on the curve
(remember the meaning of "chord" in

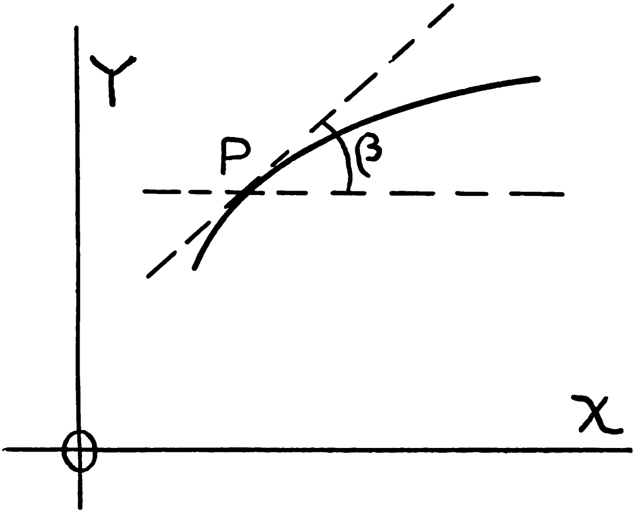
a circle, from your study of
elementary geometry?).
But if these chords were
prolonged indefinitely,
they would be called "secants"
(again like a secant of a circle).
Note that the DIRECTION of
the CHORD PQ
is the SAME as the DIRECTION of
the SECANT PQ (prolonged),
only of course the secant is
LONGER than the chord
which was prolonged to get the secant.
Note also that
the DIRECTION of
QDC, $Q_1D_1C_1$, $Q_2D_2C_2$, etc., etc.,
is always the SAME,
namely,
PERPENDICULAR to OX,
and the DIRECTION of
PD, PD_1, PD_2, etc., etc.,
is always the SAME,
namely,
PARALLEL to OX.
Thus,
as Q keeps approaching
nearer and nearer to P,
until it is finally AT P,
you will doubtless agree that:

(1) all three sides of triangle PDQ
eventually dwindle down to
ZERO in LENGTH

BUT

(2) the DIRECTION of QD and PD

has remained the SAME
throughout the process during which
Q has been approaching P,
AND
the DIRECTION of the secant PQ
has been changing so that
angle α has been changing
until FINALLY
the DIRECTION of the SECANT
has become
the DIRECTION of the TANGENT line
at P,
so that
angle α,
though it has been changing,
arrives finally at the value β,
which is the angle formed by
this tangent line at P and
a line through P parallel to OX,
thus:



And so,
although the triangle PDQ
has dwindled down to "nothing"
yet,
since the directions of
all three of its sides
are perfectly definite,
hence,
it may be enlarged to
ANY desired SIZE
without changing its SHAPE.
Thus,
if we represent
the LIMITING value of $\Delta y/\Delta x$ by dy/dx,
we now have

$$dy/dx = \tan \beta.$$

And,
though dy/dx seems to be 0/0
and therefore an "indeterminate form",*
yet,
as you saw above,
since the LIMITING value of α
is a perfectly definite value,
namely, β,
therefore dy/dx is NOT indeterminate
but is exactly equal to $\tan \beta$
thus:

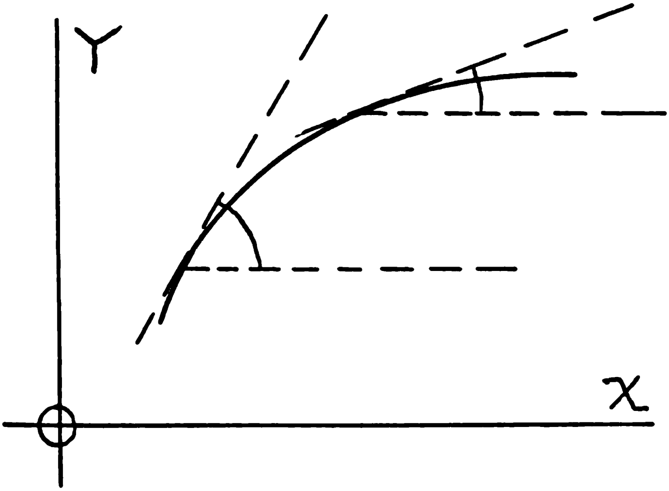
$$dy/dx = \tan \beta.$$

Of course
the value of β depends upon
the location of point P
as shown in the graph on p. 235,
angle β being larger wherever
the curve is "steeper".

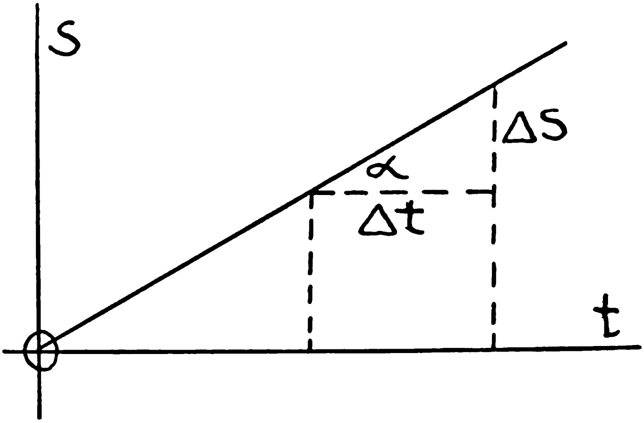
---

*See any book on Differential Calculus.

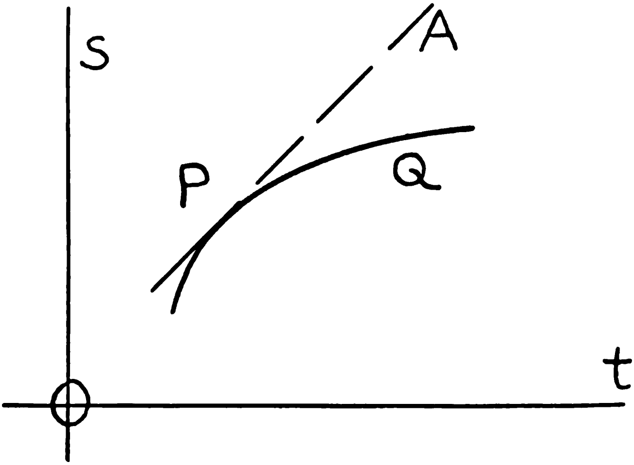
Now let us see an APPLICATION
of these ideas,
which is particularly useful in
Physics,
invented and used by
Newton and Leibnitz:
Suppose s represents distance traversed
in time t,
and suppose that $s = f(t)$ is a continuous function,



where $0 \leqq t < \infty$
and s has a definite value for
each value of t.
Then if $s = kt$
where k is a constant,
this equation represents
the straight line in
the first figure on p. 236
where $\Delta s/\Delta t = s/t = k$
is the SPEED of



the motion in question.
And you can easily see that
if $s = f(t)$ is a continuous function
(where again $0 \leqq t < \infty$
with s having a definite value for
each value of t)
does not represent a straight line
but a curve, thus:



then here the SPEED is NOT constant
but varies from point to point
on the curve,
being represented, at any point, P, by

$$ds/dt$$

which is the LIMIT of $\Delta s/\Delta t$
as Q approaches P,
as explained on p. 234.
This very important quantity, ds/dt,
is called the
"derivative of s with respect to t",*
ds and dt being called
"differentials" of s and t, respectively.
Hence the term
"Differential Calculus",
which makes possible
the study of MOTION having a
VARYING VELOCITY,
a concept much more difficult than
UNIFORM MOTION,
in which $s = rt$
where r is the simple
constant speed of motion.

Problems of UNIFORM MOTION

---

*In general,
dy/dx is called the
"derivative of y with respect to x" —
where y and x may stand for
ANY two quantities, one of which is
a function of the other —
thus y may represent the population in
a given region at time t, and
as t varies, y also varies,
and dy/dx now would represent the
RATE of CHANGE of
population with time,
etc., etc., for
other practical applications.

are solved quite easily by
elementary algebra,
as you probably know;
whereas problems of
MOTION with a CHANGING VELOCITY
require the
DIFFERENTIAL CALCULUS
which deals with
METHODS of finding the value of

$$ds/dt$$

for various KINDS of functions,
as you can see by examining
any book on Differential Calculus.
And of course you realize that
since MOTION
(not only simple UNIFORM motion!)
is of the utmost importance in
Physics,
you can see why
the Differential Calculus is
such a vital basic tool
for the solution of
many, many physical problems,
especially since,
by means of
the more general
dy/dx,
one can solve also
OTHER problems in which
one quantity is a
continuous function of another (p. 235).
But naturally you cannot
FULLY appreciate this until
you make a further study of
this wonderful subject,

or until you at least LOOK at
a book on this subject,*
even if you only read
the PROBLEMS proposed,
so you may realize
what a wide variety of questions
can be answered by means of the
Differential Calculus.

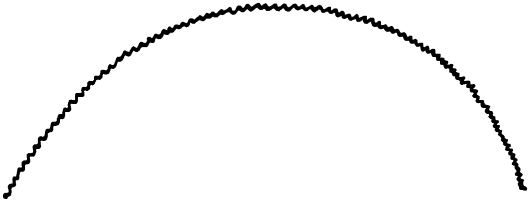
*See, for instance,
"Analytic Geometry and Calculus" by
H. B. Phillips.

19 The Calculus of Newton and Leibnitz (cont.)

As was said in the previous chapter,
the Differential Calculus shows how
to find the "slope" of a
continuous function at
any one of its points, P
("slope" meaning the tangent* of angle β,
which is formed by
the line tangent* to the curve at P
and a line parallel to OX) ;‡

*Please note that the word "tangent"
is here used in TWO different senses,
and that is indeed the
CUSTOMARY practice.
If you do not recall this from
your high school mathematics,
see p. 346 for an explanation.

‡Your "intuition" probably tells you
that any continuous curve DOES have
a definite slope at each of
its points (see diagram on p. 235).
BUT, believe it or not,
in recent times it has been realized
that a curve, even though continuous,
may be "crinkly", and may therefore
NOT have a definite slope ANYWHERE:
a "crinkly" curve is
something like this:

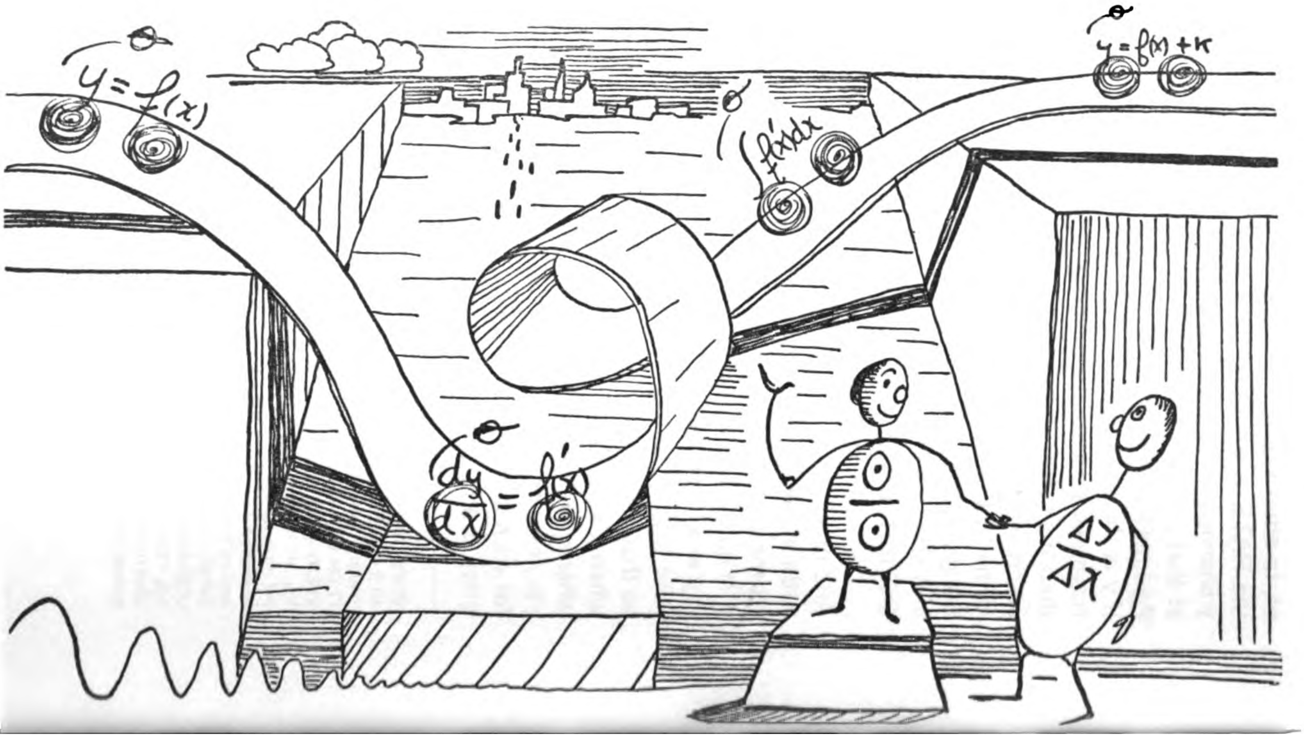


but of course this gives only a

or, in other words,
the RATE at which
y changes with respect to x
at that point, P.
And, as I said before,
if y is the distance
traversed by a point P in time x,
then the "slope" at P
(represented by dy/dx or ds/dt)
is the VELOCITY of the point at
the given instant x —
and similarly for
OTHER APPLICATIONS (p. 238).

And now,
without going into the details here
let me just add that
it is quite easy to show that
if $y = x^2$, then $dy/dx = 2x$
(see p. 270)
so that
when $x = 0$, $dy/dx = 0$; and
when $x = 1/2$, $dy/dx = 1$;
etc., etc., and
for each x in a given range
there is a definite value of dy/dx.

**VERY CRUDE idea of it,
for it is IMPOSSIBLE TO DRAW A
"crinkly" curve, since
IT CHANGES ITS DIRECTION AT
EVERY one of its points!
You will have to use your MIND,
not mere pen and paper,
to imagine it!
And so, the classical
Differential Calculus is
USELESS in such SPECIAL cases;
but, FORTUNATELY,
most functions in Physics do not lead
to such special cases.**


$y = f(x)$
$\frac{dy}{dx} = f'(x)$
$\int f'(x)dx$
$y = f(x) + k$
$\frac{0}{0}$
$\frac{\Delta y}{\Delta x}$

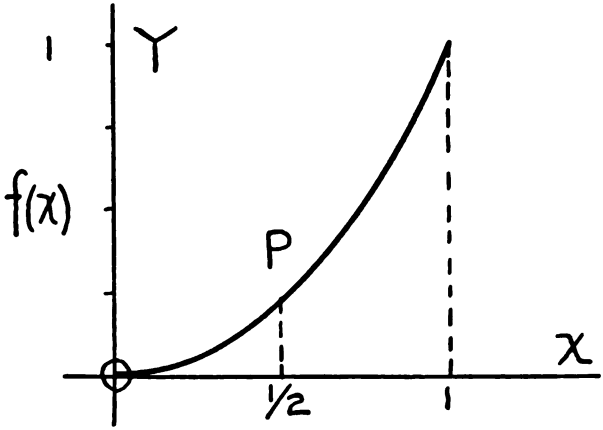
We can therefore now construct
the following table,
from which TWO different graphs
may be obtained,
as shown on p. 244.

x	$y = x^2$	$dy/dx = 2x$
0	0	0
1/2	1/4	1
1	1	2
etc.	etc.	etc.
:	:	:
:	:	:

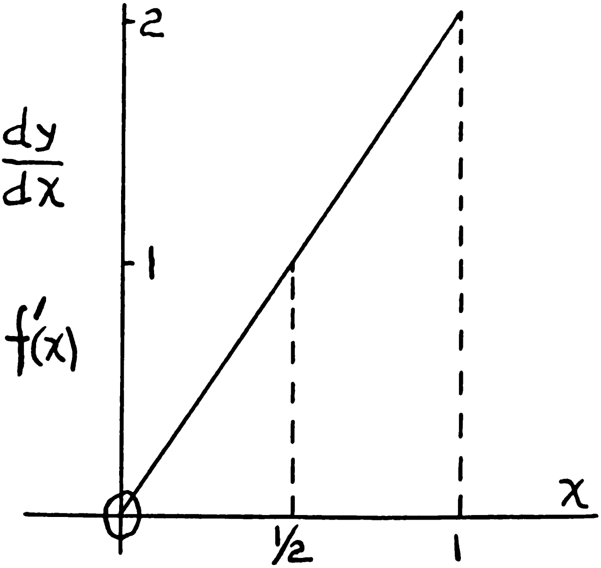
Of course you can see that
these two graphs are
RELATED to each other:
for instance,
if $x = 1/2$,
then $dy/dx = 1$ in Graph 2 is
the "slope" of the curve in Graph 1 at P,
where x is also equal to 1/2.
And, in general,
the ordinates in Graph 2 represent
the derivatives of the curve in Graph 1
at the CORRESPONDING points in Graph 1.

Now notice an interesting thing:
if $y = f(x)$ is a
GIVEN continuous function,
then,
in general,
it is possible
to find the UNIQUE value
of dy/dx, namely,
the slope of

Graph 1 ($y = f(x) = x^2$):



Graph 2 (dy/dx is another function of x thus $f'(x) = 2x$):



the tangent line at P.*
BUT note that if
the INVERSE problem is proposed,
namely,
if $dy/dx = f'(x)$ is GIVEN,
as for example in Graph 2 (p. 244)
and the corresponding $y = f(x)$ is
REQUIRED,
then you can easily see that
MANY answers are possible,
an INFINITY of them:
thus
if $y = f'(x) = 2x$ (1)
is GIVEN,
and the corresponding $y = f(x)$
(called the "primitive" of (1))
is REQUIRED,
instead of getting ONLY
the curve in Graph 1, p. 244,
we get a whole series of curves,
EACH of which
has the SAME slope for
any given value of x,
as shown in the diagram on p. 246.

Note that
when $x = 1/2$, for example,
the tangents (the dotted lines) at
P_1, P_2, P_3, P_4, P_5
are all PARALLEL to each other
so that
ALL the curves shown here
have the SAME slope at $x = 1/2$.
And, of course,

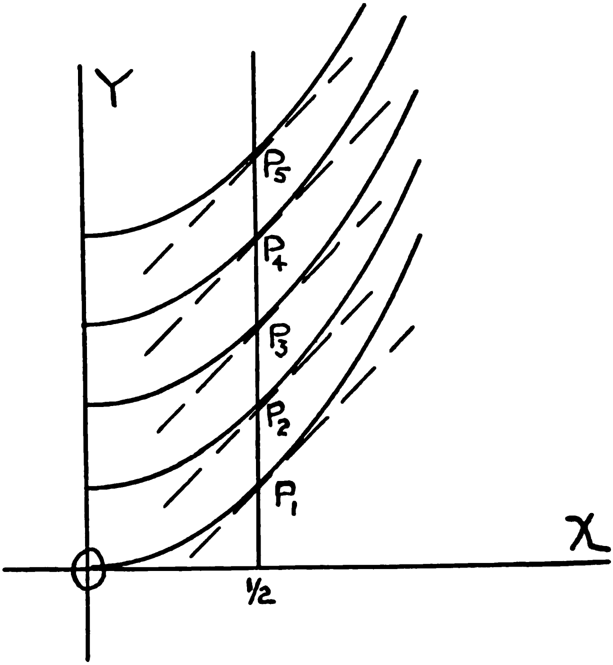
---

*Except for "crinkly"-curves
which, though continuous,
have NO definite slope
ANYWHERE (p. 240).

(1) this is true also
for EVERY value of x

and

(2) the five curves shown below
are NOT the ONLY ones that



satisfy the required conditions,
for, as you can see,
there is an infinite set
of such curves,
all having the SAME slope for
any given value of x.

Thus, for example,
if $dy/dx = f'(x) = 2x$
we may obtain:

(a) $y = f(x) = x^2$ or
(b) $y = f(x) + 1 = x^2 + 1$ or
(c) in general,
$y = f(x) + k = x^2 + k$ where
k may be any real number.

Observe that
in (a), if $x = 0$, then $y = 0$
in (b), if $x = 0$, then $y = 1$
in (c), if $x = 0$, then $y = k$
which is to say that
k is the distance on the Y axis
from O to the point where
the curve in question
crosses the Y-axis.

This process of finding
the "primitive" $y = f(x)$
FROM
the DIFFERENTIAL EQUATION $dy/dx = f'(x)$
is called
INTEGRATION
and is the subject of study of
the INTEGRAL CALCULUS* of
Newton and Leibnitz.
Note that since dy and dx
do NOT have to be regarded as

---

*Only certain simple cases of
finding the "primitive" are studied
in the Integral Calculus;
further methods of solving more complex problems
are studied in courses called
"Differential Equations" —
see any book on this subject.

being very tiny
but may be as large as you please (p. 234),
hence dy/dx is just
an ordinary fraction
and therefore
$dy/dx = f'(x)$ may be written
$dy = f'(x) \cdot dx$ and
the INTEGRAL of this is written

$$y = \int f'(x) \cdot dx$$

where the symbol $\int$
(called an "integral sign")
indicates that
an INTEGRATION is to be performed,
obtaining the primitive

$$y = f(x) + k.$$

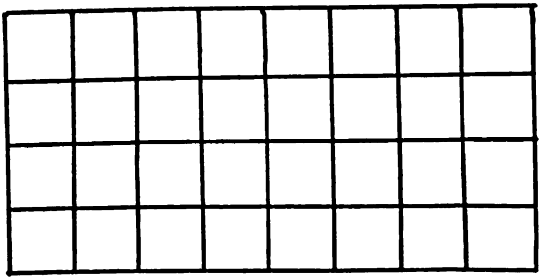
Thus you see that
DIFFERENTIATION and INTEGRATION
are INVERSE processes —
just as
ADDITION and SUBTRACTION are
INVERSE processes —
thus if you ADD 4 to 3, you get 7,
and if you now SUBTRACT the 4 from 7
you get your original 3 back again.
And similarly
DIVISION is the INVERSE of
MULTIPLICATION,
etc.
In general,
whenever you perform
a certain operation,
obtaining a certain result,
and if you then perform
another operation
upon this result

which “un-does” the original operation,
and thus land back where
you started from,
then
two such operations are
INVERSES of each other.
But of course you must remember that,
in the case of
Differentiation and Integration,
if from $y = f(x)$ you get,
by differentiation,
$dy/dx = f'(x)$,
then by integrating the latter,
you get
NOT ONLY
$y = f(x)$ back again,
but also
$y = f(x) + k$,
as explained on p. 247.
All of this,
though only briefly summarized here,
is (I hope) fairly intelligible;
but of course
to get a good working knowledge
of the subject,
as well as to appreciate
its many, many, practical applications,
would naturally require further study
of the Differential and Integral Calculus,
as well as of Differential Equations.
But now that you have
at least a bowing acquaintance
with this very valuable subject,
let us proceed.

20 Areas

Before continuing this discussion of
the Calculus of Newton and Leibnitz,
it is important to consider
the question of finding the areas
of plane figures of various shapes —
although you may NOT YET
see the connection between
this question and the
DIFFERENTIAL and INTEGRAL CALCULUS.

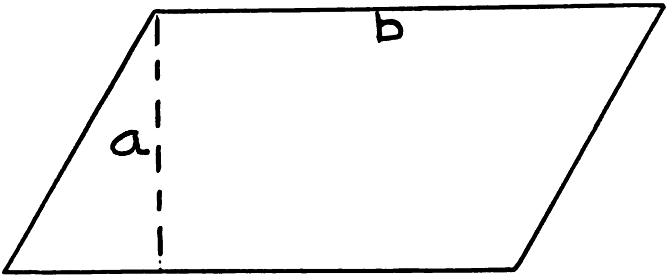
First of all,
take the simplest case,
that of finding
the area of a rectangle,
a four-sided plane figure having
its opposite sides parallel
and four right angles,
like this:



suppose it happens to be
8 inches long and 4 inches wide.
It is easy to see that

this figure can be divided up into
little squares,
each having an area of 1 sq. in.,
and that there will be
8 such squares on each row
and 4 such rows;
and therefore 32 squares in all,
so that the area of the given rectangle
is thus seen to be 32 sq. in.
And similarly, in general,
if a rectangle has dimensions a and b
then its area is
axb or ab sq. in.
Similarly, it is quite easy to show* that

(a) the area of a parallelogram
(a four-sided figure whose
opposite sides are parallel,
as in a rectangle,
but whose angles are
not necessarily right angles)



is still ab sq. in.,
where b is still the length or "base"
and a is the "altitude",

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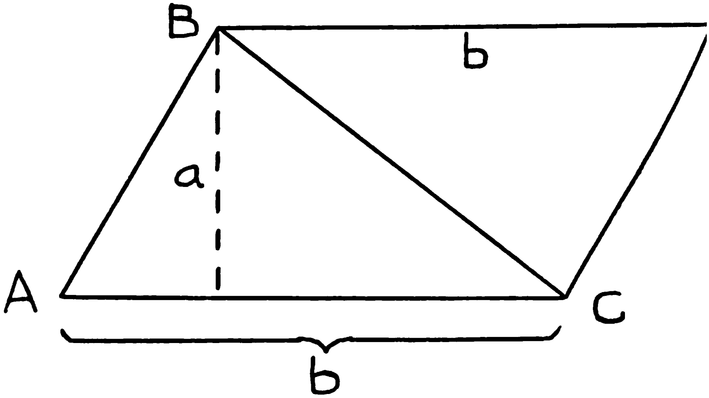
*See any book on elementary
Euclidean Plane Geometry.

which means
the PERPENDICULAR distance between
the upper and lower "bases";

(b) the area of a triangle ABC



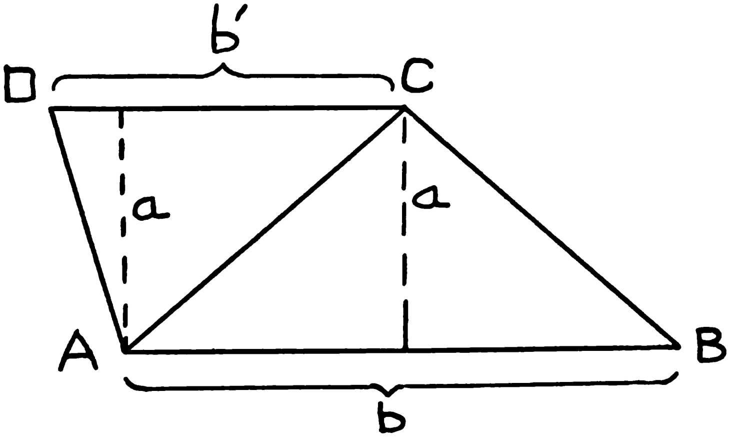
is 1/2 ab
since it is
HALF the parallelogram ab
as shown in the figure;

(c) the area of a trapezoid
(a 4-sided figure having
only ONE pair of parallel sides)
may be found
by drawing a diagonal,
thus cutting up the figure into
two triangles,
whose areas can of course be found by
the formula in (b) above:
thus the trapezoid ABCD on p. 253
(with sides AB and DC parallel)
is divided by the diagonal AC into

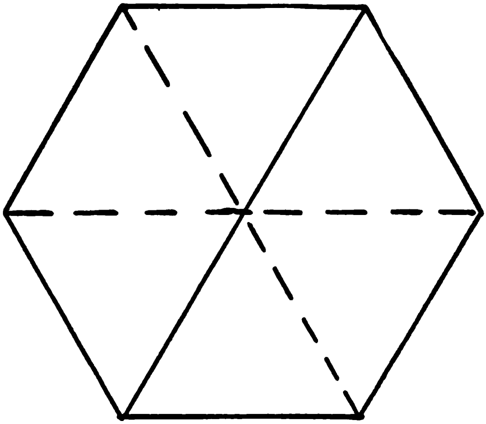
triangle ABC whose area is 1/2 ab and
triangle ACD whose area is 1/2 ab′
making the area of trapezoid ABCD
equal to 1/2 ab + 1/2 ab′
or, as you know from
elementary algebra,

$$[1/2]a(b + b').$$

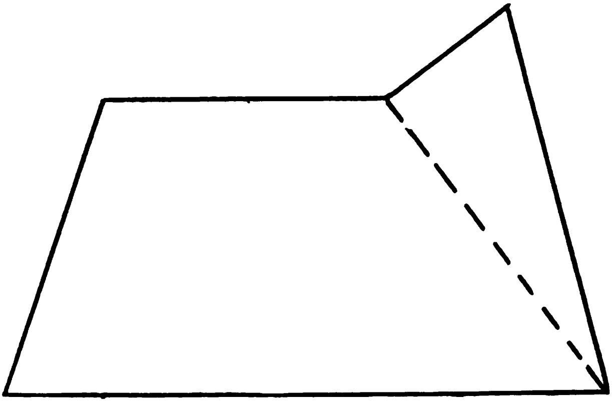
That is to say,
the area of a trapezoid is equal to
one-half its altitude times
the sum of its two bases.



And similarly,
ANY plane figure which is
bounded by straight lines
may be split up into
triangles and trapezoids,
making it quite simple to
find its area, thus:



or thus:



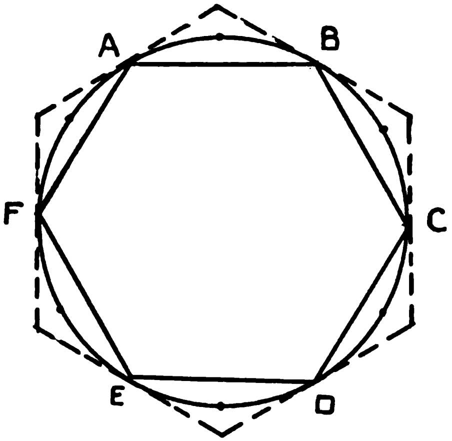
etc., etc.

So simple is this that
it is easily understood even by
children,
and it has been known for
thousands of years.
BUT
if the boundary of a plane figure is

a CURVE instead of
a series of straight lines as above,
then
the problem of finding
the area of the figure
is quite different and
much more difficult.
Let me tell you first
a little about how the
great Greek mathematicians,
about 2000 years ago,
succeeded in finding the areas of
SOME figures whose
boundaries were curves;
their methods were indeed ingenious
BUT were NOT GENERAL
and hence could not be used for
finding the area of
ANY plane figure having
a curvilinear boundary,
but were useful ONLY in
certain special cases,
as, for example, in finding
the area of a circle
(and even here the area was found only
APPROXIMATELY),
or of a portion of a parabola,
which has particular properties
making it possible to find its area.

Thus, in the case of a circle,
suppose we divide its circumference
into 6 equal parts (see p. 256),
as at A, B, C, D, E, F.
If we now join these points of division,
a hexagon is formed which is
entirely within the circle and

therefore has an area which is
LESS than the area of the circle;
and, as you already know,
the area of this hexagon
can be easily found as on p. 253.



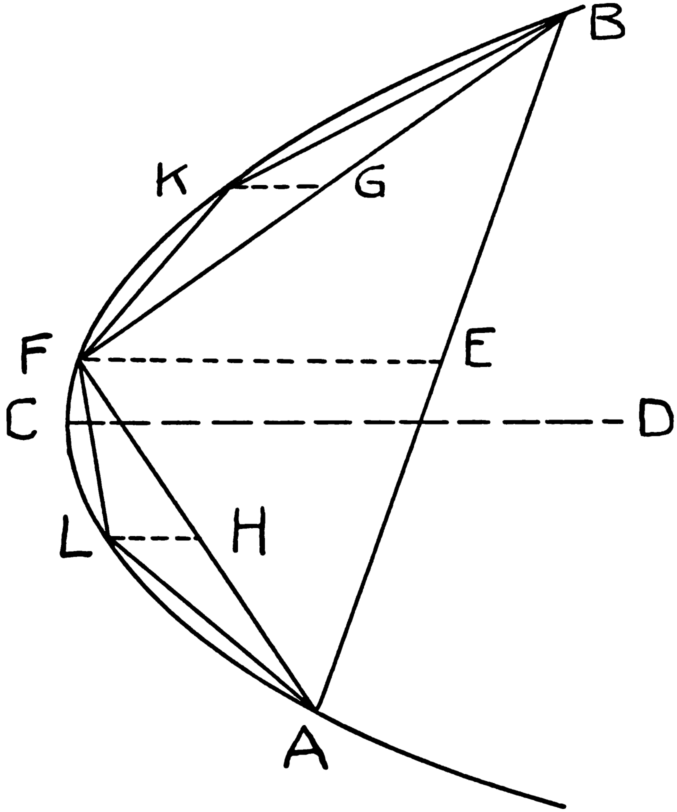
If now we draw TANGENTS at
the 6 points of division, as shown,
we again get a hexagon
(the one formed by the dotted lines),
but this hexagon is
LARGER than the circle,
and of course its area also
may be found by the method on p. 253.
And so we see that
the area of the circle is
GREATER than that of the INNER hexagon
and LESS than that of the OUTER hexagon,
and therefore
has a value somewhere between
the areas of the two hexagons,
both of which can be easily found.

Thus we already have
an APPROXIMATE value of
the area of the circle.
And you can see that
if we should divide the
circumference of the circle into
12, instead of 6, equal parts,
inserting 6 more points
half-way between A and B, B and C, etc.,
and again form
the INNER and OUTER 12-sided figures,
then both these figures are
CLOSER in size to the size of the circle
than the hexagons were —
the INNER 12-sided figure
(called a dodecagon)
being LARGER than the inner hexagon
but still LESS than the circle,
and the OUTER dodecagon being
SMALLER than the outer hexagon
but still GREATER than
the circle itself.
So now we know that
the area of the circle is
BETWEEN the areas of
the two DODECAGONS,
and we have therefore
HEMMED IN its value still further,
and thus have
a still CLOSER APPROXIMATION to
the value of the area of the circle.
And of course
if we inscribe and circumscribe
polygons having
a still larger number of sides,
we shall get
a more and more close

APPROXIMATION to
the area of the circle,
which is our goal.
As a matter of fact,
the great Greek mathematician,
Archimedes,
went as far as using
96-sided figures,
and thus found that
a circle of unit radius
has an area lying between
$3\frac{10}{71}$ and $3\frac{1}{7}$
which, expressed decimally, give
3.141 and 3.143,
thus showing that
a circle having a radius of 1 in.
has an area which is
APPROXIMATELY 3.14 sq. in.,
correct to the SECOND decimal place.
And if you should use
polygons having
even more than 96 sides,
you would get values which would be
correct to the
THIRD, FOURTH, etc., decimal places,
but even these would be
only APPROXIMATIONS,
and NOT EXACT.

Let me now show you
another problem solved by Archimedes,
the problem of finding the area of
a segment of a parabola,
namely,
the segment bounded by the chord AB
and the parabolic arc ACB
(see p. 259):

through E, the mid-point of AB,
draw a line parallel to
the axis of the parabola, CD,
and meeting the parabola at F,
forming triangle ABF.



Repeat this process by
taking the mid-points, G and H,
of the chords FB and FA, respectively,
and drawing through them
lines parallel to the axis CD,
meeting the parabola at

K and L, respectively,
thus determining the triangles
FKB and FLA.
Now it is easy to prove,
using the properties of a parabola
(see p. 351),
that the sum of
triangle FKB and triangle FLA
is equal to
1/4 of the triangle ABF;
and,
by continuing this process
still further,
that is,
by taking the mid-points of
the chords BK and KF,
you can again get two triangles
whose sum is
1/4 of triangle FKB,
and similarly
you can get two more triangles
whose sum is
1/4 of triangle FLA,
so that the total sum of
these FOUR NEW triangles
is equal to
1/4 of the sum
triangle FKB + triangle FLA,
thus making this sum of
the FOUR NEW triangles
equal to
1/4 of 1/4 of triangle ABF*
or $1/4^2$ of triangle ABF.
And if you continue this process further
you will get next

---

*Since triangle FKB + triangle FLA = 1/4 triangle ABF (see above).

a set of triangles whose sum is
$1/4^3$ of triangle ABF,
then
another set of triangles whose sum is
$1/4^4$ of triangle ABF,
and so on and so on.
Finally
the area of the
original parabolic segment
bounded by the chord AB and
the parabolic arc ACB
is approached by the TOTAL SUM of ALL these triangles,
if the above-described process is
carried on "ad infinitum",
thus, the series

$$ABF + 1/4\ ABF + 1/4^2\ ABF + 1/4^3\ ABF + \cdots$$

approaches the area of the parabolic segment.
Now this infinite series is known as
a "geometric" series
and its sum is therefore found
by means of the formula

$$S = a/(1 - r) \quad \text{(see p. 211)}.$$

Hence here

$$S = ABF/(1 - 1/4)$$

or $$S = ABF/(3/4)$$

or $$S = 4ABF/3.$$

In other words,
the area of the parabolic segment
is 4/3 as great as
the triangle ABF.
And you must admit that
the method of obtaining this result
is ingenious indeed.

And yet, as you see from p. 260,
it applies ONLY to a parabola,
and therefore does not help us at all in
finding the area of a figure
bounded by some other curve.
And now you are in a position to
APPRECIATE
the Calculus of Newton and Leibnitz
by means of which
you will soon see that
you can find the area of
ANY figure having for its boundary
ANY continuous curve.

21 The Fundamental Theorem of the Integral Calculus

Before considering this
Fundamental Theorem,
let us see the method of finding
the area of a plane figure which is
bounded by ANY continuous curve;
for you will soon realize that
this method is related to
the problem of finding
the "primitive" of a given function (pp. 245 ff.).
And it is this relationship which
constitutes the essence of
the Fundamental Theorem
referred to above.

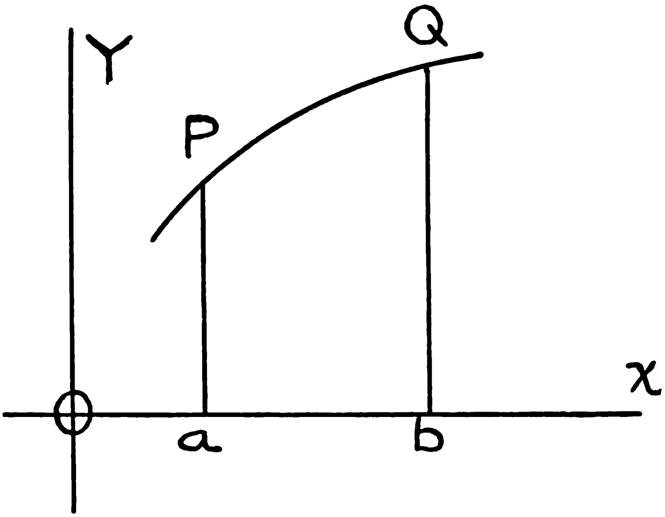
Suppose that $y = f(x)$ is a continuous function,
where x is continuous on the range

$$a \leqq x \leqq b,$$

and y has a definite value for
every value of x on this range,
thus
$y = f(x)$ is a continuous function
DEFINED in the interval ab:

$$a \leqq x \leqq b.$$

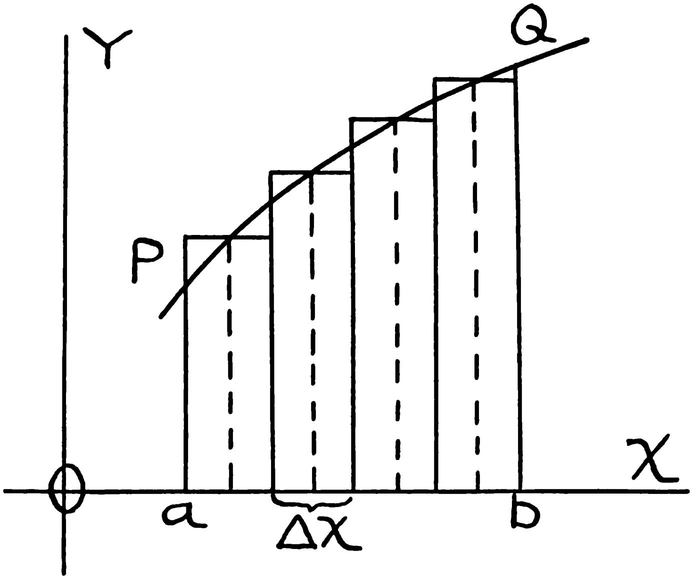
And suppose that
this function is represented in
the following graph (p. 264):



Now suppose it is required to find
the area under this curve,
namely,
the area bounded by the arc PQ,
the ordinates Pa and Qb and
the part of the X-axis ab.
You can readily see that
you would get
an approximate value of this area by
finding the sum of the rectangles
shown on p. 265.

Note that these rectangles
are formed by
dividing the distance ab into
n equal or unequal parts
(the figure happens to show n = 4
equal parts),
and, at these points of division,
erecting the sides of the rectangles

in such a way that
the ordinates represented by
dotted lines,
namely, the altitudes of the rectangles,
each lie WITHIN the various rectangles
(in the figure



each altitude happens to be
at the mid-point of
the corresponding base).

To show you how close
the approximation may be,
take the case given in this figure,
having only 4 rectangles, as shown;
let $a = 2$, $b = 3$, each $\Delta x = 1/4$,
and let the function be $y = x^2$.
Then the distance from O to

the foot of the FIRST altitude
is $2\frac{1}{8}$, is it not?
And therefore the LENGTH of
the FIRST altitude is $(2\frac{1}{8})^2$ or 289/64,
since $y = x^2$ for any point on the curve.
Similarly
the SECOND altitude is $(2\frac{3}{8})^2$ or 361/64,
the THIRD is $(2\frac{5}{8})^2$ or 441/64
and the FOURTH is $(2\frac{7}{8})^2$ or 529/64.
Hence the respective AREAS of
the 4 rectangles are:

(1) $289/64 \cdot 1/4 = 289/256$
(2) $361/64 \cdot 1/4 = 361/256$
(3) $441/64 \cdot 1/4 = 441/256$
(4) $529/64 \cdot 1/4 = 529/256$

(since the area of a rectangle is
equal to the product of
its base and altitude).
Therefore the SUM of all FOUR rectangles
is
$(289 + 361 + 441 + 529)/256 = 1620/256$
or 6.328 (to the nearest 3rd decimal place),
whereas,
the EXACT answer
(found by the method described below)
is $6\frac{1}{3}$, which
to the nearest 3rd decimal place, is
6.333
which differs from 6.328 only by
LESS than 1/10 of 1%!

Now of course you will admit that
if you split ab into many MORE parts,
making MORE rectangles,
each of which is narrower than

those shown on p. 265,
the sum total of all these
narrower rectangles
will be still nearer
the desired area under the curve.
And, finally,
if you let all the Δx's approach zero,
and thus the number, n, of rectangles
approach infinity,
then
the LIMIT of their sum will equal
the area in question.
Thus we have the relationship:
Area under the curve =

$$f(\xi_1)\cdot\Delta x_1 + f(\xi_2)\cdot\Delta x_2 + \cdots$$

where ξ is a Greek letter
(pronounced ksi)
and ξ_1 is the value of x WITHIN the
FIRST rectangle
where its altitude is erected,
and of course $f(\xi_1)$ is
the LENGTH of this altitude,
and so on for ξ_2 etc., etc.
Or, writing this infinite series
MORE BRIEFLY,
it becomes

$$A = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(\xi_i)\cdot\Delta x_i$$

which is read as follows:
"The area under the curve is equal to
the LIMIT (as Δx approaches 0) of
the sum, $\sum$*, of

---

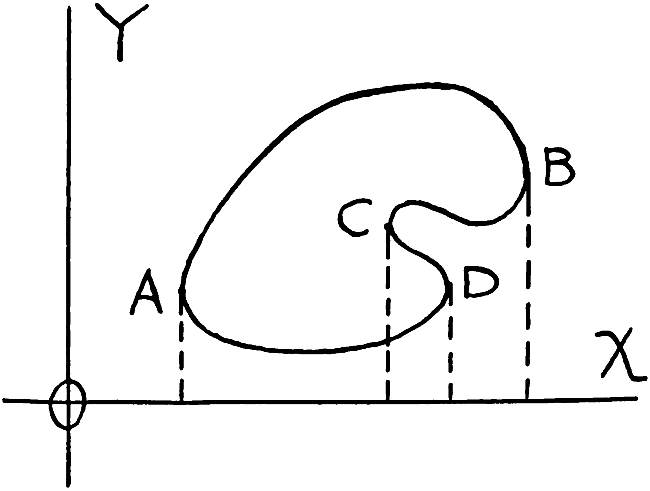
*Σ is another Greek letter, "sigma",
equivalent to our s, and hence
appropriate to stand for "sum".

all terms of the form

$$f(\xi_i) \cdot \Delta x_i$$

each of which is the area of
one of the rectangles
as i takes on all positive integral values
from 1 to n."

Similarly,
if an area is completely bounded by
a continuous curve
without any straight lines at all,
like this:

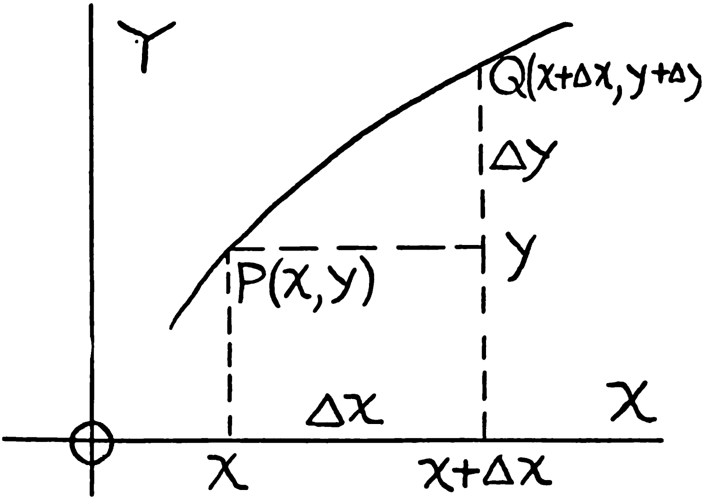


the area WITHIN the curve
may be found by the same method
by adding and subtracting
certain areas as follows:
(area under AB) — (area under BC) +
(area under CD) — (area under AD),
each of which can be found by

the method described above.

Let us now find
the area under a curve in another way,
and, by comparing the two methods,
thus obtain
the Fundamental Theorem of
the Integral Calculus,
promised above.

But first
let me remind you of the definition of
the "derivative", dy/dx, (p. 237),
though in a slightly different form:



If $y = f(x)$ (1)
then $y + \Delta y = f(x + \Delta x)$. (2)
Subtracting (1) from (2), we get
$\Delta y = f(x + \Delta x) - f(x)$.
Now, dividing by Δx, this becomes

$\Delta y/\Delta x = [f(x + \Delta x) - f(x)]/\Delta x.$
And now,
as Q approaches P,
$\Delta y/\Delta x$ approaches dy/dx,
so that

$$dy/dx = \lim_{\Delta x \to 0} [f(x + \Delta x) - f(x)]/\Delta x. \qquad (3)$$

And, as on p. 244,
dy/dx is itself a function of x
and may be represented by $f'(x)$,
and (3) may be written:

$$f'(x) = \lim_{\Delta x \to 0} [f(x + \Delta x) - f(x)]/\Delta x. \qquad (4)$$

And, as was said before (p. 247),
$f'(x)$ is the derivative of $f(x)$,
and conversely,
$f(x)$ is the "primitive" or
"anti-derivative" of $f'(x)$.
Before continuing this discussion
let us linger for a moment
to show how to find out that
$f'(x) = 2x$ when
$f(x) = x^2$ is given (see p. 241),
by applying (4) above to
this special case:
since $f(x) = x^2$, then

$$f(x + \Delta x) = (x + \Delta x)^2 = x^2 + 2x \cdot \Delta x + (\Delta x)^2$$

$$\therefore\ f'(x) = \lim_{\Delta x \to 0} [x^2 + 2x \cdot \Delta x + (\Delta x)^2 - x^2]/\Delta x$$

$$\therefore\ f'(x) = \lim_{\Delta x \to 0} [2x \cdot \Delta x + (\Delta x)^2]/\Delta x$$

or $f'(x) = \lim_{\Delta x \to 0} (2x + \Delta x)$;
and therefore,
as Δx approaches 0, we get
$f'(x) = 2x$
or $dy/dx = 2x$.

Thus, by means of (4),

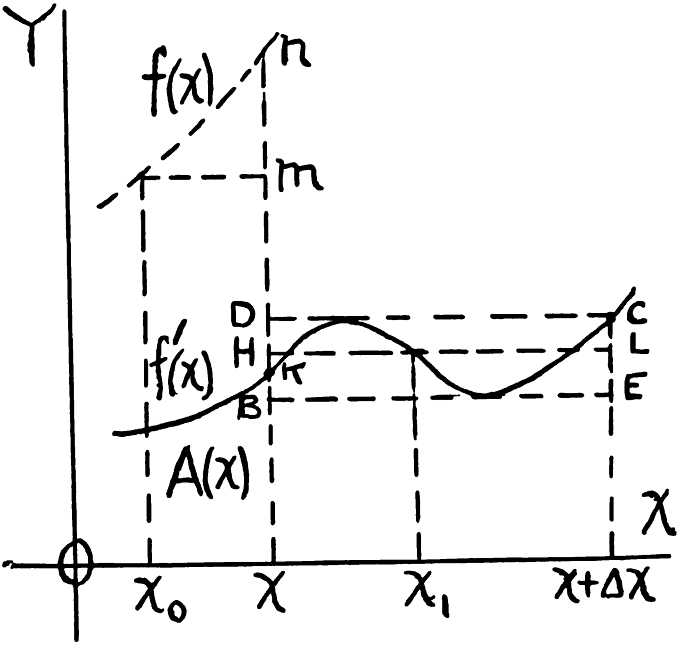
from DIFFERENTIAL CALCULUS
we can find dy/dx for
any function $y = f(x)$.
And, conversely,
given $f'(x)$,
we can find the primitive or
anti-derivative $f(x)$
by means of the
INTEGRAL CALCULUS,
as has already been said.

And now let us return to
the discussion of the above-mentioned
Fundamental Theorem.
Let us therefore find
the area under a curve
by another method
(as mentioned on p. 269).
Consider the diagram on p. 272
in which
$f'(x)$ is the derivative of $f(x)$
and hence $f(x)$ is
the anti-derivative of $f'(x)$
as on p. 244).
The area under the portion of the curve KC
is obviously equal to

$$A(x + \Delta x) - A(x).$$

This is evidently LESS than
the area under DC and
GREATER than the area under BE,
and is therefore equal to the area of
some rectangle between these two,
as, for example,
the area under HL
and is therefore equal to $f'(x_1) \cdot \Delta x$.
And so we have

$A(x + \Delta x) - A(x) = f'(x_1) \cdot \Delta x$
or
$[A(x + \Delta x) - A(x)]/\Delta x = f'(x_1)$.
And, as Δx approaches zero,
we get
$\lim_{\Delta x \to 0} [A(x + \Delta x) - A(x)]/\Delta x = f'(x)$



since x_1 is BETWEEN x and $x + \Delta x$.
Consequently, as on p. 270,
$A(x)$ must be
an anti-derivative of $f'(x)$.
But
$f(x)$ is GIVEN as
an anti-derivative of $f'(x)$ (p. 271).
Hence

$$A(x) = f(x) + k \text{ (pp. 247 ff.)}. \qquad (5)$$

Now, when $x = x_0$,
then $A(x_0) = 0$ (see diagram p. 272)
and (5) becomes

$$0 = f(x_0) + k.$$

Therefore $k = -f(x_0)$.
Substituting this value of k in (5)
it becomes
$A(x) = f(x) - f(x_0)$
which is equal to the distance mn
in the diagram on p. 272.
In short,
the number of square units in
the area $A(x)$
is equal to
the number of linear units in
the distance mn,
and this latter may be written

$$\int_{x_0}^{x} f'(x)\,dx$$

which is read:
the definite integral of $f'(x) \cdot dx$
"from x_0 to x",
and is equal to $f(x) - f(x_0)$.
It is called a "definite" integral
because it has a definite value
for specific values of
the "upper and lower limits" shown
on the integral sign, $\int$,
whereas
an indefinite integral has no such
designations on the integral sign
and refers to
an infinite set of different primitives
(see p. 246).
FOR INSTANCE:

the indefinite integral $\int 2x\,dx$
is equal to $x^2 + k$
where k may have various values (p. 247),
whereas
the definite integral $\int_1^2 2x \cdot dx$
is equal to $(x^2)_1^2 = 2^2 - 1^2 = 4 - 1 = 3$,
a definite value,
namely, the area under the curve
$y = x^2$ from $x = 1$ to $x = 2$.
And so, finally, we come to the
Fundamental Theorem of
the Integral Calculus by
equating the two values of
the area under a curve:

(1) the one given on p. 267:

$$A = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f'(\xi_i) \cdot \Delta x_i{}^*$$

which was obtained by finding
the LIMIT of the SUM of
a lot of rectangles
and

(2) the one given on p. 273:

$$A = \int_{x_0}^{x} f'(x) \cdot dx$$

which involves the process of
INTEGRATION, which is
the INVERSE of
DIFFERENTIATION (p. 248)
and has really nothing to do with
adding up rectangles!

---

*Note that I am writing f′ here
because the A in question, on p. 272,
is the area under the f′ curve.

But once you realize that
(1) and (2) above
both give the area under a curve,
so that

$$\lim_{\Delta x \to 0} \sum_{i=1}^{n} f'(\xi_i) \cdot \Delta x_i = \int_{x_0}^{x} f'(x) \cdot dx;$$

and, since the function is
the same on both sides,
it is simpler to write
f instead of f′,
thus obtaining:

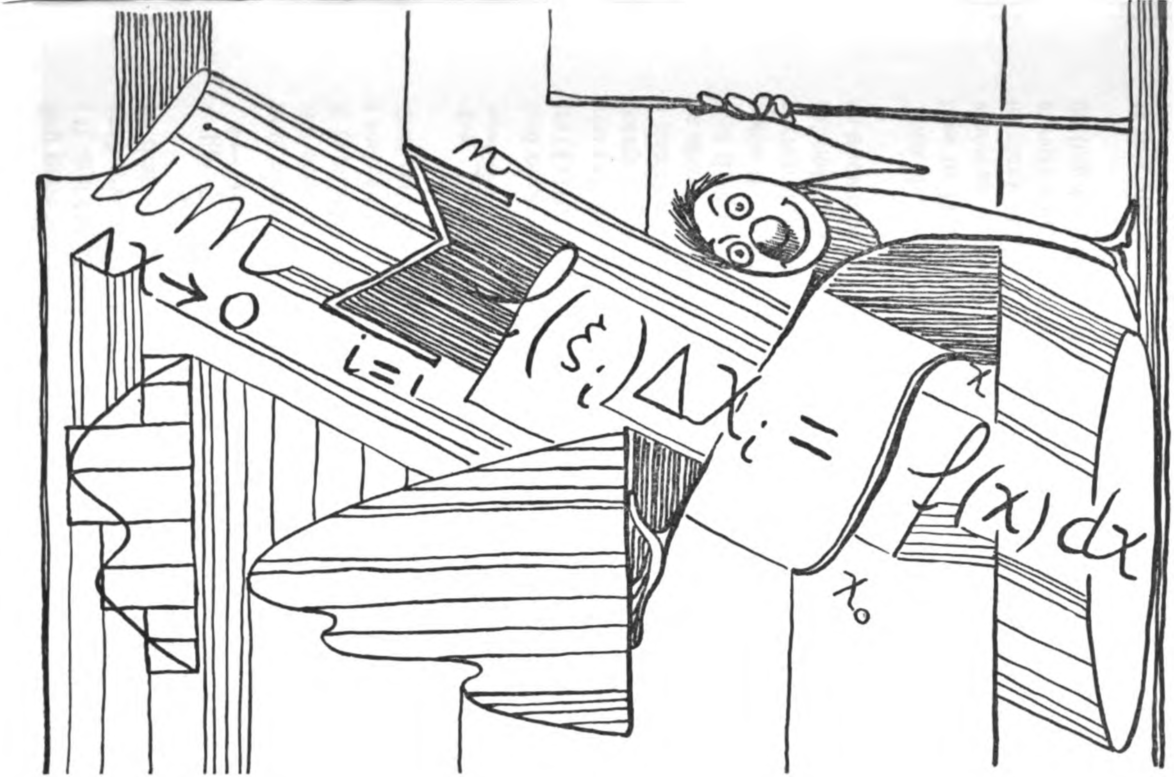
$$\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(\xi_i) \cdot \Delta x_i = \int_{x_0}^{x} f(x) \cdot dx \ldots . \qquad (6)$$

and once you learn how to
INTEGRATE various functions,
you can then tackle
many practical problems by
a judicious "slicing" up
as shown on the left side of (6),
and then proceed to
solve the problem by
performing the necessary integration
(almost automatically),
as shown on the right of (6).

Thus the left side of (6) shows you
how to THINK of the problem,
whereas the right side of (6) has
in many cases
almost mechanical methods of
getting the answer.

Is this all Greek to you?!

Well, do not worry too much,



$\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \int_{x_0}^{x} f(x)\,dx$

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for at least you have SEEN
ONE application of the Calculus,
namely,
the solution of the problem of
finding the area
enclosed by a continuous curve.
Of course you would appreciate this more
if you had been racking your brains
for years
in trying to find such areas,
as some people have!
Let me add, however, that,
by means of the Calculus,
one can find
not only areas,
but a host of other
important things in
Physics,
as you can at least see if
you merely look at some of the
problems proposed in a book on
the INTEGRAL CALCULUS,
just as I suggested in the case of
the DIFFERENTIAL CALCULUS (p. 239).

But, needless to say,
if you really want
a WORKING knowledge of
these two
wonderful mathematical tools,
you must roll up your sleeves
and WORK at it,
just as you would have to
get out and really PLAY golf,
if you wanted to improve your game.
But, if you stand on the side-lines,
you need not become a good golfer if

you wish merely to
watch others play and just to
have a good time
appreciating the game.
And of course
the same is true here:
if you have followed
the few things I have told you
in this little book,
and then even glance at
some of the problems in
a book on Calculus,*
you can have yourself some fun
without becoming a professional.

And now let us see
how these tools have been
further developed in modern times,
and you will surely realize that
man, if he properly uses
the SAM within him,
is truly a marvellous creature,
instead of the
greedy, thievin', ulcerous
good-for-nothin'
he can become
when he allows himself
to degenerate into an
ANTI-SAMITE!

---

*There is a particularly good selection
of practical problems
at the end of each chapter in
"Analytic Geometry and Calculus" by
H. B. Phillips
(formerly of the
Massachusetts Institute of Technology).

22 MODERN Integration

You have already seen how
the area under a continuous curve
may be found by means of
the Integral Calculus of
Newton and Leibnitz.
And, by the same means,
you can even calculate
the volumes of solids,
as well as many other
important practical things,
as you can easily see by
looking this up in
any book on Calculus.*

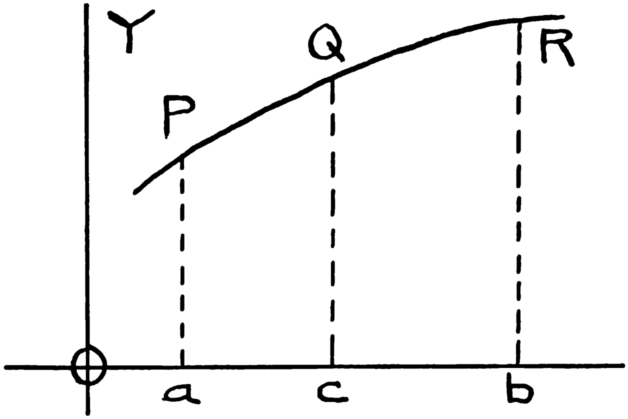
It follows from
the definition of
"definite integral" (see Ch. 21)
that
for a continuous function $y = f(x)$,
where x has the range $a \leqq x \leqq b$
and y is defined in this range,
if $x = c$ lies between $x = a$ and $x = b$,
then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

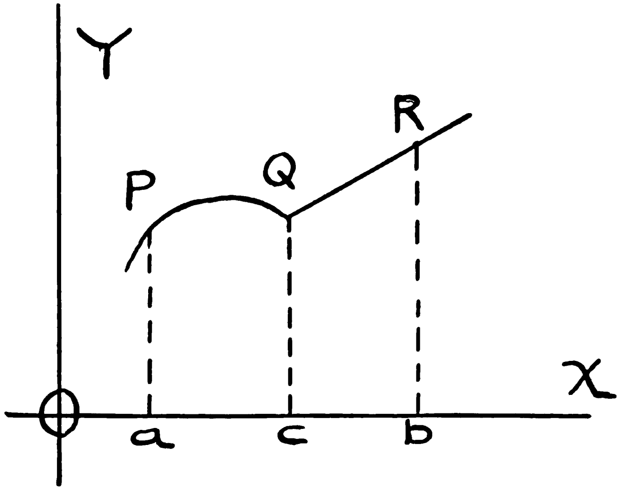
thus:

---

*As, for example,
"Analytic Geometry and Calculus" by
H. B. Phillips,
pp. 88 to 93, et al.



which is to say that
the area under PR is
equal to the sum of
the areas under PQ and QR —
and of course this would hold
even if PQ and QR are
two DIFFERENT curves, thus:



where PQ may be an arc of a parabola
and QR a segment of a straight line,
etc.
Now, it sometimes happens in
PRACTICAL APPLICATIONS,
that it is necessary to find
the integral of a function which is
NOT continuous,
in which cases of course
the Calculus of Newton and Leibnitz
are useless!
And hence
various NEW KINDS of INTEGRALS
have been developed in MODERN times
by Riemann, Stieltjes, Lebesgue, Denjoy,
known as
the Riemann Integral,
the Stieltjes Integral,
the Lebesgue Integral,
etc.,
respectively.

This is of course not the place
to give a detailed account of
these marvellous modern tools,*
but
perhaps I can give you some notion of
these modern types of integration

---

*See "Survey of the Theory of Integration" by
the great mathematician,
Jesse Douglas, —
a paper written for
The Galois Institute of Mathematics at
Long Island University,
in which Douglas gives
a brief but masterly review of
Modern Integration,
together with
a bibliography for
the more serious student.

with the aid of
the theory of sets of points
of which you have acquired some idea
from the previous chapters of
this little book.

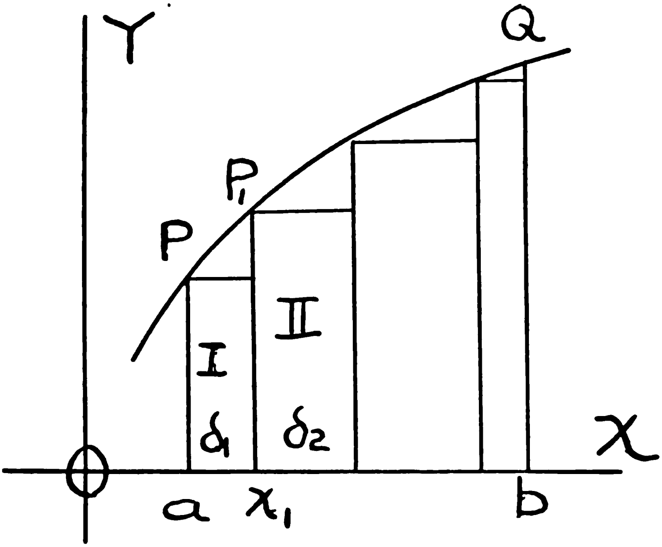
Let us go back for a moment to
ordinary Integration:
you will remember (p. 265) that
in order to find
the area under an arc of
a continuous function $y = f(x)$
where the range of x is given by

$$a \leqq x \leqq b,$$

(1) we first divided up the distance ab
into n (equal or unequal) parts,
called a "partition" of
the interval ab;
(2) constructed rectangles on these
sub-intervals;
(3) erected altitudes somewhere WITHIN
each rectangle;
(4) calculated the area of
each of these rectangles;
and then
(5) summed the areas of
all the rectangles so obtained;

thus getting
an APPROXIMATE value of
the area desired,
namely,
that under the curve.
Now,
if, instead of constructing
the rectangles

as shown on p. 265,
with the dotted lines as altitudes
located WITHIN the rectangles,
suppose we had constructed
the rectangles thus:



using as altitude in each rectangle
the LEFT-hand ordinate:
thus for rectangle I, the altitude is aP,
for rectangle II, the altitude is x_1P_1,
etc., etc.,
then you can easily see that
the sum of THESE rectangles
will surely be LESS than
the required area under the curve PQ.
Let us represent this sum by

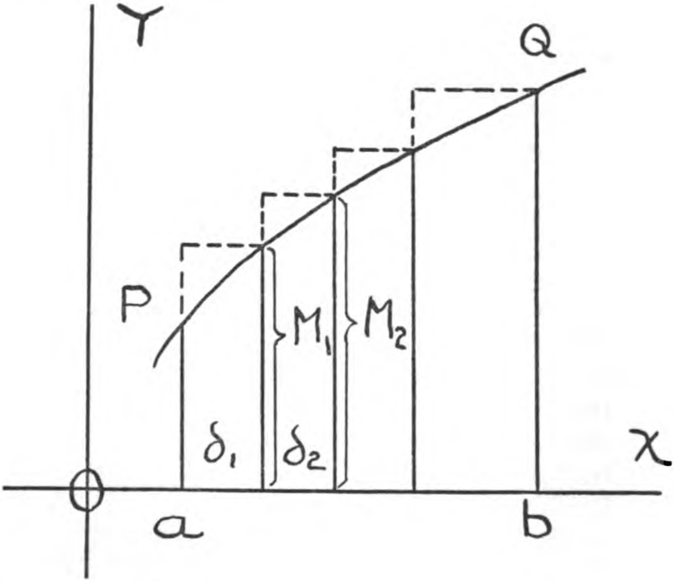
$$s = \sum_{k=1}^{n} m_k \cdot \delta_k$$

where each m_k is the
SMALLEST ordinate in the
kth sub-interval, δ_k.
And of course
if you make a DIFFERENT partition of ab,
and get a correspondingly
DIFFERENT set of m_k's,
you will obviously get
a DIFFERENT value of s.
And, as you already know,
if you now let n approach ∞,
with every δ_k approaching zero,
s will approach a LIMIT,
namely,
the desired area under the curve itself.
Similarly,
if, for a certain chosen
partition of ab,
you constructed the rectangles
in such a way that
the altitudes would be the
LARGEST ordinate, M_k,
(instead of the SMALLEST)
in each sub-interval,
as shown in the diagram on p. 285,
the sum of the rectangles would now be
GREATER than the desired area under PQ,
would it not?
Representing it by

$$S = \sum_{k=1}^{n} M_k \cdot \delta_k$$

and again allowing
n to approach ∞
and every δ_k to approach zero,
S would approach a LIMIT,
which would again be

the desired area under the curve itself.
In short,
if $y = f(x)$ is
a continuous curve,
as is the case with functions treated in
the ordinary Calculus of
Newton and Leibnitz,



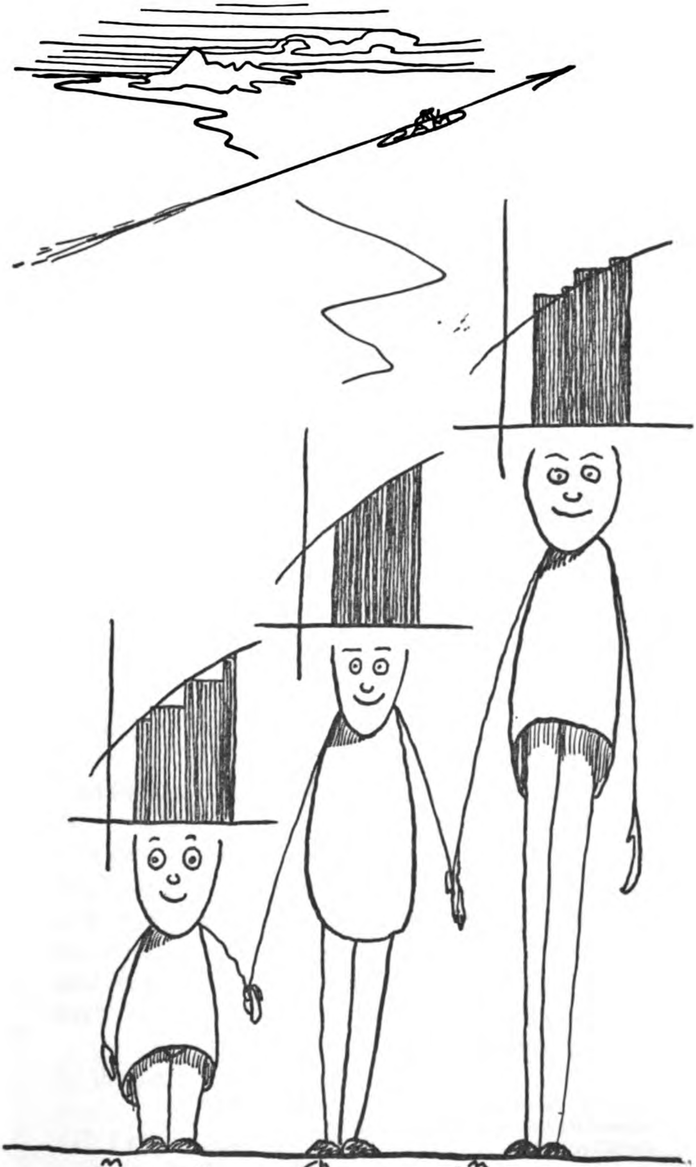
the limits of s and S are
both the SAME,
each being equal to
the desired area under PQ,
and each being equal to
the definite integral

$$\int_a^b f(x) \cdot dx$$

as was said above;

and the same value is obtained
regardless of what partition of ab
is chosen
and regardless of
which altitudes are chosen!
I hope you appreciate that
this is a truly wonderful result!

But now suppose that
y = f(x) is NOT a continuous function
in the interval, ab, in question,
then
the above-mentioned wonderful result
no longer necessarily applies,
and yet such functions
DO arise in
scientific work.
And so the question
"What to do?"
became an urgent one.
It was then that
Riemann undertook to extend
Newton's Calculus to
DISCONTINUOUS functions;
and he found that
the above-described processes of
finding the LIMITS of s and S
could still be employed
BUT that
for DISCONTINUOUS functions
these two limits were
NOT ALWAYS IDENTICAL!
When they ARE IDENTICAL,
the function is said to be
"Riemann integrable" or
"integrable R" for short —
otherwise not.



$$\sum_{k=1}^{n} m_k \delta_k < \int_a^b f(x)\,dx < \sum_{k=1}^{n} M_k \delta_k$$

And then of course
new questions arose:

(1) how many discontinuities
may a function have and
still be integrable R?
and

(2) what to do about
those discontinuous functions which
are NOT integrable R
and which still arise in
scientific work?

In answer to Question (1),
let me say that
it has been shown that
EVERY function
bounded* in an interval ab,
and having a FINITE number of
points of discontinuity
IS integrable R.
Furthermore,
a bounded function may have even
a DENUMERABLE‡ INFINITY of discontinuities
and STILL be integrable R.
And,
I shall even show you a function,
due to Georg Cantor himself,
which has
a NON-denumerable‡ infinity of
discontinuities and yet
is ALSO integrable R.

---

*"Bounded" means that the LARGEST and the SMALLEST values of the function in the interval ab are BOTH FINITE.

‡See Chapter 23.

In the next chapter
I shall give you the general criterion for
a function to be integrable R.
This does not imply, however,
that
ALL functions are integrable R!
For example,
let me now show you a function,
due to Dirichlet,
which, though bounded,
is NOT integrable R:
thus consider the function, $\phi(x)$,
which has the value 1 when
x is rational,
and the value 0 when
x is irrational;
here $m_k = 0$ (see p. 284)
and $M_k = 1$ (p. 284);
hence
the LIMIT of s, as δ_k approaches 0,
is 0,
and
the LIMIT of S is 1.
Therefore,
since these two limits are
NOT identical,
this function is
NOT integrable R (see p. 286)
although it is bounded
since its smallest value is 0,
and its largest is 1,
both finite numbers.

In the next chapter, then,

(a) I shall show you the function
promised on p. 288;

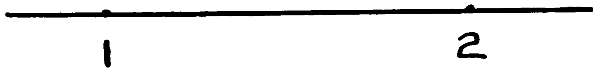
(b) I shall give you,
in very compact form,
the criterion, due to Lebesgue,
for judging,
IN GENERAL,
what kind of function
is integrable R —
for which you will have to know
the meaning of
the very important concept of
the "measure" of a set of points;

(c) and, we shall return to
the consideration of
question (2) on p. 288.

23 Modern Integration (cont.)

In order to keep the promises made
in (a), (b), and (c) at the end of
the last chapter,
let us first consider the meaning of
the term "measure" of a set.
You already know that
a set of points has
a cardinal "number"
as, for example,
the set of real points in
the interval $0 \leqq x \leqq 1$
has the cardinal number, c (p. 111);
and of course
this cardinal, c,
is NOT the same as
the LENGTH of this interval,
which is of course 1.
Thus, the two intervals
$0 \leqq x \leqq 1$ and $0 \leqq x \leqq 2$
each have the
SAME CARDINAL NUMBER, c (p. 111),
whereas
the LENGTHS of these two intervals
are DIFFERENT,
being 1 and 2, respectively.
Now,
the "measure" of a set is
NOT the same as its cardinal number
but is a different property of a set,
explained as follows:

Suppose, for example,
the set happens to contain
a finite number of points,
say, x = 1 and x = 2;
in this case
it is possible to cover
the points of the set by
a single interval, thus:



This interval may of course have
various lengths, and still be able to
cover the 2 points of this simple set.
Also,
the 2 points of this set
may be covered by
2 separate intervals, thus:

having a combined total length which
again may have various values.
Or,
the points of our set
may be covered by
two overlapping intervals
(this is hard to show in a diagram
but very easy to imagine)
which again may be of
various total lengths.
Now, if you consider
all possible ways of covering
the points of this set,

you can easily see that
there will be among them
one having
the shortest possible length:
in our simple case above
this shortest possible total length is
of course zero if
you use two non-overlapping intervals
and think of them as becoming
shorter and shorter
(that is, $< \epsilon$, where ϵ is
a positive quantity chosen
as small as you please)
until they are only just long enough
for each to cover
one of the two points —
the length of each such interval
then being zero,
since a point itself has zero length,
and hence also
the shortest total length of
BOTH intervals is zero.
And, you can easily see that
the same would be true if
you had, instead of two,
ANY finite number of points.
We then say that
the "content" of a set of
any finite number of points is
zero.
Consider now
the set of rational points
(call the set E)
in the interval ab,
and cover this set, E, with
a finite number of sub-intervals:
$b_1 - a_1$, $b_2 - a_2$, etc., etc.;

then,
if you represent this covering by
the letter j,
and let $l(j)$ represent
the total length of all
these sub-intervals,
we may write

$$l(j) = \sum_{n=1}^{n} (b_n - a_n)$$

which is read:
"The total length of this covering
is equal to the sum ($\sum$) of
all the sub-intervals
$b_1 - a_1, b_2 - a_2, \ldots\ldots$
as n varies
from 1 to some finite number, n."
And
the least possible length of
all such possible coverings
is called
the lower bound of these total lengths
and is represented by

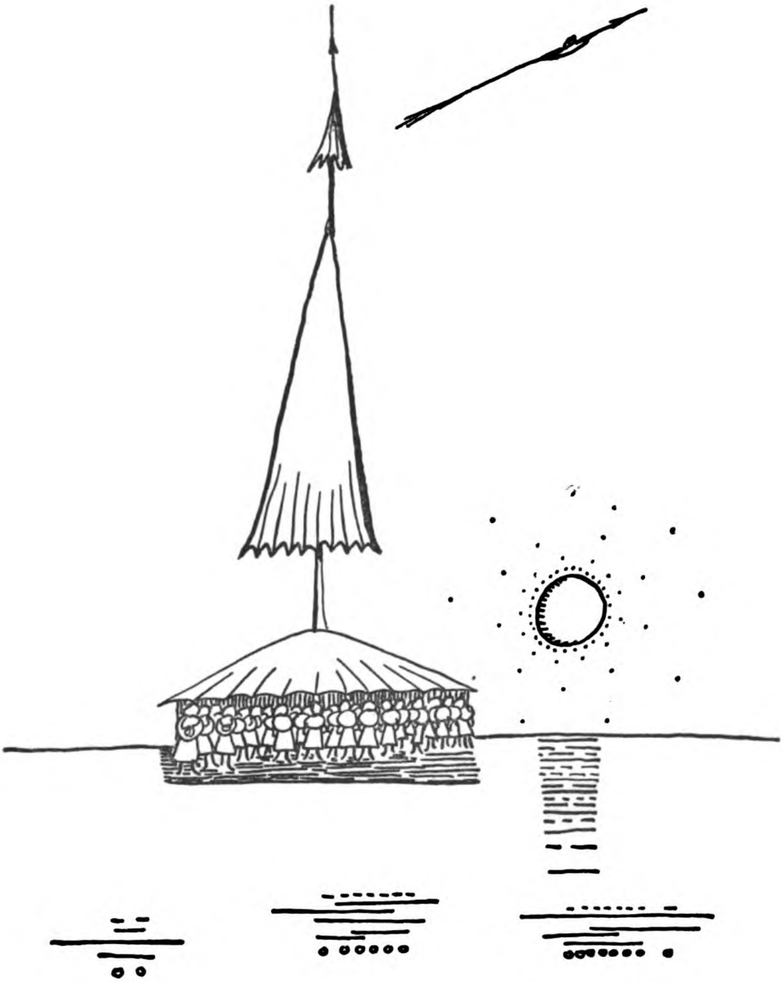
$$\underline{B}l(j).$$

Now this lower bound is called
the "outer content" of the set E,
which we designate by

$$\bar{c}(E);$$

thus $\bar{c}(E) = \underline{B}l(j) = \underline{B} \sum_{n=1}^{n} (b_n - a_n).$

Consider next
the set of points in the interval ab
which are NOT in E
(i.e. which are NOT rational points
and hence are



the irrational points in ab,
meaning of course
the points whose distances from O
are irrational numbers) ;
represent this new set by CE
(where C stands for the word
"complement" —
thus the set CE is
the complement of the set E
in the interval ab).
Of course CE also has
an "outer content"
represented by $\bar{c}(CE)$.
And if we next consider
the LENGTH of the interval ab,
namely, $b - a$,
and form $(b - a) - \bar{c}(CE)$
this result is called
the "inner content" of
the original set, E,
and is represented by $\underline{c}(E)$;
thus

$$\underline{c}(E) = (b - a) - \bar{c}(CE).$$

Now,

$$\bar{c}(E) \geqq \underline{c}(E)$$

that is,
the outer content of a set is
GREATER THAN OR EQUAL TO
its inner content;
and when these
HAPPEN to be EQUAL,
that is,
WHEN

$$\bar{c}(E) = \underline{c}(E)$$

then we say that
the set E

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HAS a CONTENT,
represented by c(E);
thus

$$c(E) = \bar{c}(E) = \underline{c}(E) \quad \text{IF} \quad \bar{c}(E) = \underline{c}(E).$$

Finally,
if the covering contains
a DENUMERABLE infinity of sub-intervals
(instead of a finite number, as above),
and if we represent this new covering by
J (instead of j as above)
then,
instead of a "content",
we use the term
"MEASURE".
Thus

$$l(J) = \sum_{n=1}^{\infty} (b_n - a_n)$$

and here n takes on
all positive integral values from
1 to ∞.
And we now have

$$\overline{m}(E) = \underline{B}l(J)$$

that is,
the "outer measure" of E is equal to
the lower bound of l(J)
for all possible covering systems of E.
And, as before,
the "inner measure" of E,
represented by $\underline{m}(E)$,
is given by

$$\underline{m}(E) = (b - a) - \overline{m}(CE) \qquad (1)$$

where, again,
CE represents the set which is
complementary to E in ab,

namely,
those points of ab which
are NOT in E.

Again, as with "content"

$$\overline{m}(E) \geqq \underline{m}(E),$$

and,
when $\overline{m}(E)$
HAPPENS TO BE EQUAL TO $\underline{m}(E)$
we say that the set E
HAS a MEASURE,
designated by $m(E)$,
thus

$$m(E) = \overline{m}(E) = \underline{m}(E) \quad \text{IF} \quad \overline{m}(E) = \underline{m}(E).$$

And, finally,
since the class of covering systems j
(finite)
is part of
the class of covering systems J
(denumerably infinite)
hence $\overline{c}(E) \geqq \overline{m}(E)$
and therefore

$$\overline{c}(E) \geqq \overline{m}(E) \geqq \underline{m}(E) \geqq \underline{c}(E),$$

from which we see that
if a set HAS a CONTENT
(i.e. if $\overline{c} = \underline{c}$)
then it surely HAS a MEASURE
(for then it would follow that $\overline{m} = \underline{m}$),
but not conversely.

Are you getting a little dizzy?
If so,
let me steady your nerves by
showing how all this

applies to our illustration above,
namely,
the set, E, of rational numbers
in the interval ab,
where $0 \leqq x \leqq 1$.
As you already know
this set contains
a DENUMERABLE infinity of
numbers or points,
and we may therefore
represent these points by
$x_1, x_2, \ldots\ldots x_n, \ldots\ldots$
Now,
if we can cover x_1 by an interval
of length $\epsilon/2$
(where ϵ is a positive quantity which
may be chosen as small as you please),
in such a way that x_1 is
right in the middle of the interval $\epsilon/2$
so that this interval extends from

$$x_1 - 1/2(\epsilon/2) \text{ to } x_1 + 1/2(\epsilon/2)\,;$$

and similarly cover x_2 by
an interval of length $\epsilon/2^2$,
so that it extends from

$$x_2 - 1/2(\epsilon/2^2) \text{ to } x_2 + 1/2(\epsilon/2^2)\,;$$

and so on for all the other points,
so that, in general,
the point x_n is covered by
an interval of length $\epsilon/2^n$
so that it extends from

$$x_n - 1/2(\epsilon/2^n) \text{ to } x_n + 1/2(\epsilon/2^n).$$

And now,
adding all these intervals together

we get the total length

$$\epsilon/2 + \epsilon/2^2 + \cdots + \epsilon/2^n + \cdots$$

which, as you see,
is a geometric progression
whose sum is

$$S = (\epsilon/2)/(1 - 1/2)$$

or $$S = (\epsilon/2)/(1/2)$$

which is equal to ϵ;
and since ϵ may be taken
as small as you please
it therefore approaches zero.
Thus the outer measure, $\overline{m}(E)$ of
the above-mentioned set of
rational points, E, is
zero;
and since $\overline{m}(E) \geqq \underline{m}(E)$ (p. 298),
hence its inner measure, $\underline{m}(E)$, is
also zero,
and consequently
this set HAS a measure (p. 298),
namely,
ZERO.
And so, in equation (1) on p. 297,
since $\underline{m}(E) = 0$ and $b - a = 1$,
therefore $\overline{m}(CE)$ must equal 1,
that is,
the outer measure of
the complementary set,
namely,
the set of IRRATIONAL numbers in
the interval ab where $0 \leqq x \leqq 1$
is equal to 1.

I can now redeem
the promises (a), (b), and (c)

made at the end of
Chapter 22.
Let me begin first with (b):
this can now be told,
very compactly,
as follows:

A necessary and sufficient condition
that a bounded function $f(x)$,

$$a \leqq x \leqq b,$$

be Riemann integrable
is that
the points of discontinuity of $f(x)$
form a set E of
ZERO MEASURE.

Thus you now see that
whereas
Newton and Leibnitz showed how
to integrate CONTINUOUS functions,
we now know also
how to integrate
SOME kinds of DISCONTINUOUS functions,
namely those which are
"Riemann integrable",
that is,
those DISCONTINUOUS functions
whose discontinuities form
a set of zero measure.

And now,
let me redeem promise (a) at
the end of the last chapter,
by showing you a set which
contains a NON-denumerable set of
discontinuities and

still has ZERO MEASURE
and IS therefore
still INTEGRABLE R.
This set,
due to Georg Cantor himself,
consists of
THOSE real numbers in the interval

$$0 \leqq x \leqq 1$$

which can be expressed in
the radix-system based
on the number 3 (p. 131)
expressed as follows:

$$x = a_1/3 + a_2/3^2 + \cdots + a_n/3^n + \cdots \qquad (2)$$

where the digits $a_1, a_2, \ldots, a_n, \ldots$
are RESTRICTED to the values 0 and 2
(the digit 1 being excluded);
in other words,
the restricted set in question (call it E)
does NOT include ALL the real numbers in
the interval $0 \leqq x \leqq 1$
(represented by (2))
but only those restricted as
above-mentioned.
In the first place,
it is easy to see that
even this restricted set is
NON-denumerable and therefore
has the cardinal number, c:
for,
since each a_n may take on
two possible values,
namely, 0 and 2,
and since there is
a DENUMERABLE infinity ($\aleph_0$)
of a's here,

hence x may take on
$2^{\aleph_0}$ different values;
but $2^{\aleph_0} = c$ (see p. 133);
hence the cardinal number of this set
is c,
and it is therefore
a NON-denumerable set.
And now let me show you that
it still has zero measure:
consider first the numbers

$$a_1/3 + a_2/3^2 + \cdots + a_{n-1}/3^{n-1} + 0/3^n + 2/3^{n+1} + 2/3^{n+2} + \cdots \qquad (3)$$

and

$$\frac{a_1}{3} + \frac{a_2}{3^2} + \cdots + \frac{a_{n-1}}{3^{n-1}} + \frac{2}{3^n} \qquad (4)$$

Obviously all points of (2) which
lie BETWEEN (3) and (4)
do NOT belong to the
RESTRICTED set E (p. 302).

As you see,
the nth a in (3) has the specific value 0
and all subsequent a's each have
the specific value 2;
you can easily see that
the "specific" part of the series (3)
can easily be summed as follows:
since its first term is $0/3^n$
whose value is 0,
it may be disregarded in the summation;
furthermore,
all the following terms have
the common factor, 2,
so that the sum may be written

$$2[1/(3^{n+1}) + 1/(3^{n+2}) + \cdots] \qquad (5)$$

and obviously
the part inside the square brackets
is a geometric series whose sum is

$$\frac{1}{3^{n+1}} \div (1 - 1/3)$$

or

$(1/3^{n+1})/(2/3)$

or

$1/(3^{n+1}) \cdot 3/2$

or

$(1/2) \cdot (1/3^n)$.

Hence the required sum in (5) above
is equal to $1/3^n$.
Substituting this value in (3) on p. 303,
it becomes

$$a_1/3 + a_2/3^2 + \cdots + a_{n-1}/3^{n-1} + 1/3^n. \tag{6}$$

Hence all points
within the interval (4) – (6) or

$$(a_1/3 + a_2/3^2 + \cdots + a_{n-1}/3^{n-1} + 2/3^n) -$$
$$(a_1/3 + a_2/3^2 + \cdots + a_{n-1}/3^{n-1} + 1/3^n)$$

do NOT belong to our
chosen restricted set, E, on p. 302,
and therefore
DO belong to the complementary set CE
since (2) on p. 302 represents ALL
the real numbers in the given interval;
note also that
the LENGTH of this interval is $1/3^n$
since all terms containing the a's
cancel out,

and we have left only

$$[2/(3^n)] - 1/3^n$$

which is of course equal to $1/3^n$.
And since each a may have
two values, 0 and 2, as stated above,
and since there are
$(n - 1)$ such a's,
hence there are
2^{n-1} such intervals,
and therefore their TOTAL length is

$$\sum_{n=1}^{\infty} (2^{n-1})/(3^n)$$

which is the same as
$(2^0/3) + (2/3^2) + (2^2/3^3) + \cdots$
which is a geometric progression having
the common ratio 2/3,
so that its sum is
$(2^0/3)/(1 - 2/3)$
or $(1/3)/(1/3)$
or 1.
Thus the content of this
complementary set (CE) is 1;
and since the length of
the entire interval chosen on p. 302
is also 1
(i.e. $b - a = 1 - 0 = 1$),
therefore
the content of set E
being $(b - a) - c(CE)$
is $1 - 1$
or zero.
And hence
(as stated on p. 298)
its measure is also zero
and it is therefore "integrable R"!

Thus
Riemann integration
is capable of handling
DISCONTINUOUS FUNCTIONS HAVING
not only a finite number of
discontinuities
but also a denumerable infinity of
discontinuities
and even, in some cases,
a NON-denumerable infinity of
discontinuities,
a feat which
could NOT be accomplished
by means of the classical integration of
Newton and Leibnitz,
powerful though that tool was!

And so you see that
the tools of mathematics,
as well as those of
the physical world,
are becoming more and more
streamlined and powerful
in MODERN times.

Let me now, in the next chapter,
say just a few words about
(c) on p. 290
which referred to the question of
what has been done about
those discontinuous functions
which are NOT integrable R
and yet arise in scientific work.

24 Concluding Chapter on Modern Integration

All this modern integration
discussed in the last two chapters
has been developed since
the late 90's and early years of
this 20th century —
and has been found to be
extremely useful.
And yet,
college students
(even those majoring in
mathematics or science)
have, for the most part,
not even heard of it!
Indeed,
they have not heard of any of the
progress in mathematics since
about 1800!
Thus they do not know that
there are now
many algebras,
geometries,
logics —
they know nothing of
transfinites,
etc., etc.
They are still studying
ONLY
Analytic Geometry and Calculus,
which date back to
the 17th century!
Can you imagine

a physics major who
has never seen
or even heard of
an automobile,
a radio,
an airplane,
etc., etc.?!
Is it not high time that
the courses in mathematics,
both in college and high school,
be revised so as to give to students
at least a bird's-eye view
of these modern developments
in mathematics?!

And so
let us continue with
the important subject of
modern integration,
even if you get only
an inkling of what it is about.

In a function $y = f(x)$,
whether continuous or discontinuous,
x is usually called
the independent variable,
and y the dependent variable.
If now
the independent variable is
some function of x
instead of x itself,
say $\alpha(x)$,
and if we then have to find
the definite integral

$$\int_a^b f(x) \cdot d\alpha(x)$$

instead of

$$\int_a^b f(x) \cdot dx$$

as before,
it can still be done,
under certain conditions,
and this is called
the Stieljes Integral
since it was introduced by
T. J. Stieljes in 1894;
for the necessary conditions
mentioned above
see
"Survey of the Theory of Integration"
by Jesse Douglas,
referred to previously (p. 281).

And now a few useful remarks,
and a brief summary:

(1) the MEASURE of an INTERVAL is
its LENGTH;

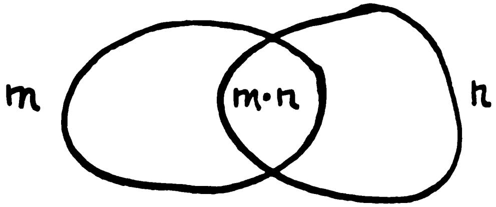
(2) the MEASURE of a POINT is
ZERO;

(3) starting with points and intervals,
we may form
the "logical SUM"* of any such,
their "logical PRODUCT"* or

---

*The "logical SUM" of
any two CLASSES of objects,
as, for example, two sets of points,
each of these sets being
represented by one of the closed curves
m or n in the diagram at the foot of p. 310 —
consists of all objects (points) contained in
m or n or their intersection;
and the "logical PRODUCT" of m and n

the complement of any one of them,
any finite or denumerably infinite
number of times
in succession,
and these results will still
each have a MEASURE;
such sets are called
"constructible"
and are said to be
"Borel measurable" or
"measurable B"
(because they were considered by
the mathematician Borel),
and include practically all sets
used in analysis.
But the definition of "measure"
given in Chapter 24,
and due to Lebesgue,
is broader than this and
includes more kinds of functions,
so that
all sets which are measurable
according to Lebesgue
are "measurable B",
but not vice versa.



is ONLY their intersection, mn.
For a simple introduction to
the ALGEBRA OF CLASSES
see my little book
"Mits, Wits, and Logic".

(4) any bounded function, f(x),

$$a \leqq x \leqq b,$$

is integrable R IF
the points of discontinuity of f(x)
form a set of ZERO MEASURE.

(5) any bounded function
which HAS a MEASURE,
WHATEVER the value of this
MEASURE,
is integrable in
the sense of Lebesgue.

(6) another important distinction
between
integration according to Riemann and
integration according to Lebesgue
is that in the FORMER
the partition is made in
the interval ab of
the INDEPENDENT variable
(or so-called "argument"),
whereas
in Lebesgue Integration
the partition is made in
the interval of the FUNCTION f(x).

(7) From (4) and (5) above, it follows
that neither
the Riemann nor the Lebesgue
process of integration
is sufficiently powerful to convert
EVERY finite-valued function F′(x)
BACK to its primitive F(x).*

*See p. 42 of "Survey of the Theory of Integration"
by Jesse Douglas, referred to on p. 281.

And so we come finally to
a process of integration which
IS adequate to solve
this problem in
the remaining cases:
this was invented by Denjoy
(his papers on this subject
appeared in the years 1912-1917)
and was called by him
"totalization";*
but for the details of this process
see H. Kestelman's
"Modern Theories of Integration" (1937).
For further bibliography
on the subject of
Modern Integration
I refer you again to
the excellent article by
Jesse Douglas
to which I have referred
several times.*

Thus the idea of
"actual infinity",—
as, for example,
a denumerable infinity ($\aleph_0$),
a non-denumerable infinity (c),—
is basically important in
the subject of
modern integration
(see Chs. 22-24),
which in turn
is basically important for
the solution of
many practical problems.

---

*See again the above-mentioned paper
by Jesse Douglas.

And consequently
not to approve of
the concept of "actual infinity",
is to lose
the invaluable aid of
many modern tools in
the solution of
many important problems.
"But", you might say,
"why speak of
'approving' or 'disapproving' of
a branch of mathematics?!"
To which the answer is
that
the subject of "actual infinity"
has been a CONTROVERSIAL one
because,
in the course of its development,
various CONTRADICTIONS have arisen,
and, as you know,
a CONTRADICTION in mathematics is
like an EARTHQUAKE,
shaking the foundations of
a beautiful and useful structure!

And now that
you have had a glimpse of
this beautiful and useful structure,
you must be anxious to know

(a) what are the contradictions
which have arisen?

(b) what has been done about them?

(c) what is the present status of
the whole subject?

These three questions
will be discussed in
the remaining chapters of
this little book,
and I hope that
when you have finished reading it
you will come out with
a great respect for the
DIGNITY OF MAN,
who,
when he is at his best,
can and does
accomplish most marvellous things
by means of the
SAM within him —
that is, by means of
his powers of observation (S),
his intuition (A),
and his ability to reason (M);
but I hope also that
you will realize that
he is NOT GOD,
that he is fallible,
that NOTHING he does is PERFECT;
and that
whatever he constructs
he must continue to watch and care for,
that he must always be on the alert
lest a wolf in sheep's clothing is
hidden in the structure,
that the price of
continuing to build and enjoy
the structure
is
ETERNAL VIGILANCE.

And I hope also that

you will realize that these are
NOT MERE WORDS,
and that
the subject of mathematics,
because of its specificity,
is an excellent domain
by means of which
you can really appreciate
MAN'S GREATNESS
as well as his
LIMITATIONS.

I believe you can realize this
even if you are
NOT a professional
mathematician or scientist,
for you will surely be aware of
the GREAT HUMAN DRAMA in
the accomplishments of
mathematics,
rather than looking at it as
a device for
mere boring calculations!
You see
MACHINES can do
your CALCULATING for you,
but it takes
a human SOUL,
a human IMAGINATION,
a human MIND,
to understand and appreciate
what mathematics really is.

Perhaps this little book
may be of some help in
giving you this attitude towards
mathematics.

25 Paradoxes

In the first place,
let us consider
what IS a paradox?
Surely it is not anything in "S",
for in "S" we OBSERVE
this, that, and the other thing,
and it is not until
we THINK about our observations
that we may encounter
"paradoxes".
In other words,
a paradox can arise
IF a certain observation
is in contradiction with
any of our "postulates"
(conscious or unconscious),
and "postulates" are of course
the basis for thinking in any system.
A few illustrations
will make this clearer:

(1) Take first the "paradox":
    the WHOLE may be EQUAL to
    one of its PARTS (see p. 85) —
    sometimes referred to as
    Galileo's paradox.
    This is a "paradox" ONLY IF
    you are dealing with
    FINITE quantities,
    for which
    the WHOLE is always GREATER than
    any of its PARTS,

as you were taught in
Euclidean Geometry,
where you have
FINITE lengths of lines
or angles of FINITE size, etc.

On the other hand,
when you wish to think about
TRANSFINITES
("actual" infinities),
the idea that
"the WHOLE may be EQUAL to
one of its PARTS"
becomes perfectly acceptable,
as you saw on p. 84;
in fact in the
Theory of Transfinites
which you have been reading about
in this little book,
it is one of the basic postulates and
is NOT a "paradox" at all,
because in this system
you do NOT have
the OTHER postulate that
"the WHOLE is ALWAYS GREATER than
any of its PARTS" which
applies ONLY to FINITE sets,
as I said above.
Thus,
for DIFFERENT systems of thought,
you have
DIFFERENT postulate sets,
just as in DIFFERENT GAMES
you have
DIFFERENT sets of rules,—
the rules for chess are not the same as
the rules for baseball!

And,
it is only WITHIN a system of thought
that you must not have
contradictions or "paradoxes"—
just as WITHIN a certain game,
you must not have
contradictory rules!
So you already see that
a PARADOX has to do with
a set of rules or postulates
rather than with "S".

But now let me give you

(2) another illustration,
in which a contradiction arose
which SEEMED to be in
the observations themselves (!),
but
since Einstein understood so well
the nature of "paradox",
he proceeded to examine
the POSTULATE SET of
classical physics,
and found that
if a change in viewpoint toward
TIME
were taken as a NEW postulate,
then the "paradox" in question
would be
and indeed was
eliminated!

Of course I cannot here
go into the details of
The Einstein Theory of Relativity;*

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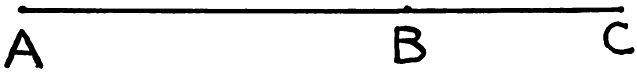
*But see Part I (the Special Theory)

let me merely tell you
what the paradox was:
certain laboratory EXPERIMENTS ("S")
showed that
the distance from A to B



was 186,000 miles,
whereas
OTHER laboratory EXPERIMENTS (also "S")
showed that
the distance from A to C
was 186,000 miles!
You can readily see that
this was an intolerable situation,
and you can see that
it seemed that various
"experimental facts"
contradicted each other!
But of course, as I said above,
the trouble was NOT in
the observations themselves,
but rather in
the way of THINKING about them,
and Einstein's idea that
TIME is RELATIVE to the observer
did the trick in
straightening out the situation
and eliminating this unbearable
PARADOX.

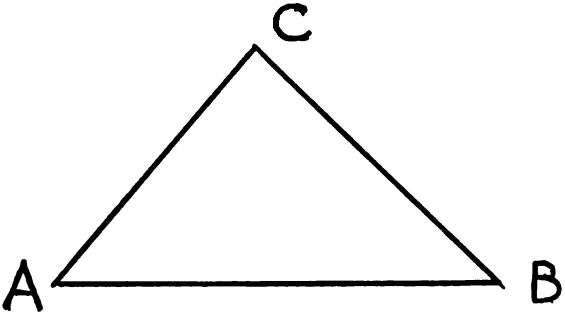
---

of my little book on
The Einstein Theory of Relativity,
where I believe you will
get the point quite easily.

Of course I do not expect you,
from this much-too-brief description,
to really understand
the issue and the solution
(for this you will need
to look up the matter in
the little book referred to
in the foot-note on p. 320) —
but at least you can already see
that the situation described on p. 320
IS paradoxical,
and you can also see that
the paradox lay in the way of
THINKING
about the situation,
just as in the paradox
described in (1) on p. 316 —
and this is the main point that
I am trying to put across to you
at this time.

Before going into
the paradoxes
which were discovered in
Cantor's Theory of Transfinites,
as well as
the way in which they have been
eliminated —
let me give you

(3) one more illustration from
Euclidean Geometry.
Believe it or not,
if you use the postulate set
originally given by
Euclid himself,
you can start with a triangle, ABC,



in which CA $\neq$ CB
and yet PROVE that
CA = CB!

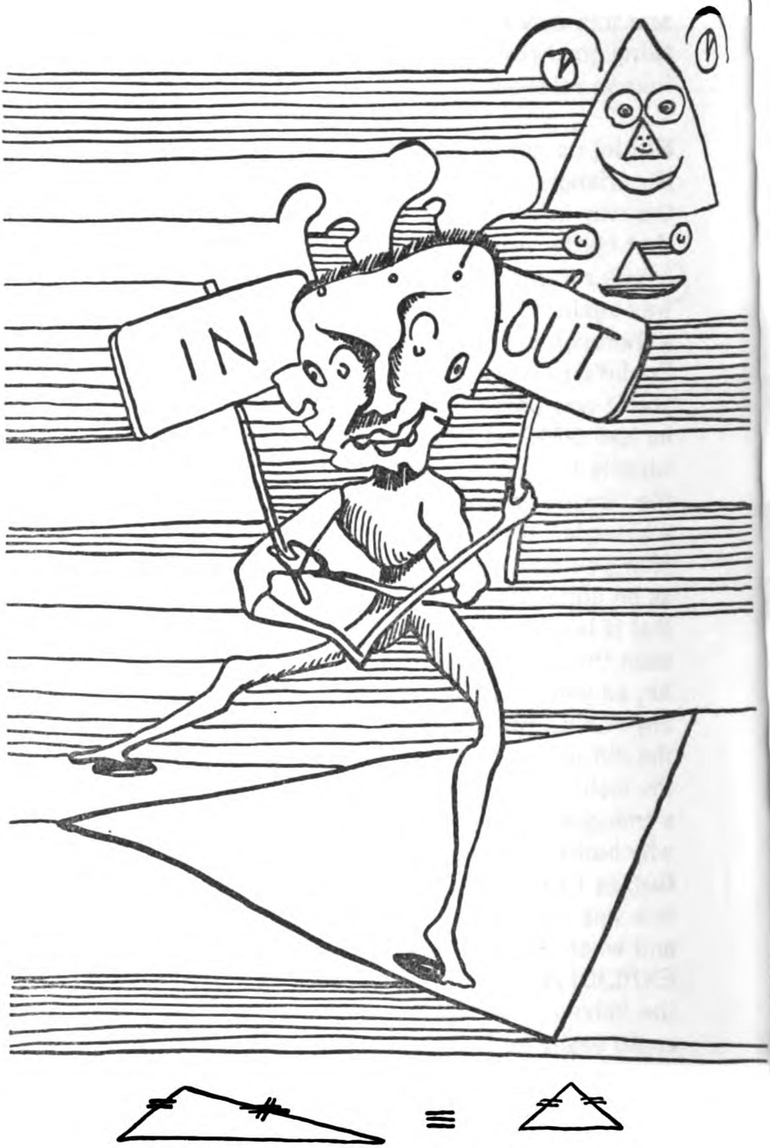
Again,
I cannot make too many detours here
by going into the details of this
PROOF,*
for all I am trying to do here is
to impress you with the fact that
whenever a PARADOX arises,
what is needed is
a thorough examination of
the way in which we are
THINKING.
Perhaps if we fully realize this
we may stop throwing the blame
for our paradoxical situations
on all the wrong things,
but rather
look into our own
basic THINKING about the matter —

*But again you can easily
acquaint yourself with it by
looking it up in my
"The Education of T. C. Mits",
where you will find the necessary details.

and this may even
bring good results in
human relations et al.

But let us now go back to
the triangle on p. 322:
the way in which
that matter was straightened out
(quite recently, by the way!),
was again
a thorough examination of
Euclid's own postulate set;
and it was found that
he had OMITTED to mention
what is to be meant by
the "inside" and the "outside" of
a triangle.
Now you may think,
as he quite naturally did,
that it is unnecessary to
state this distinction explicitly,
for, as you might say,
any FOOL can so easily see
the difference between
the inside and the outside of
a triangle, —
why bother to explain it?
But, as I said,
this was indeed the difficulty,
and when the distinction was made
EXPLICITLY
the "phoney" PROOF
could easily be ruled out!

Perhaps you might say that
if THINKING is to be that DELICATE,
where things that are


IN
OUT
≡

perfectly "obvious"
have to be explained and emphasized,
you might become afraid
to talk at all,
or even think,
about anything!

Well, in the first place,
it might not do too much harm if
a good deal of the "blah"
were really eliminated —
I heard someone say recently:
"It is too bad that
the art of conversation is dying out,
but not the power of speech!"
A good wise-crack this,
but much too drastic, don't you think?
In other words,
we do not really wish to
give up our power of speech!
But
we certainly need to curb it
with our SAM,
and to do more thinking,
DELICATE thinking,
before turning on that
IRRESPONSIBLE flow of blah.
And for this
we have to learn
WHAT THINKING MEANS,
and that is why
mathematics is REALLY important,
for in mathematics we find
probably the best SAMPLE of
the HUMAN MIND in action.
It is so useless for anyone to say
"Put your thinking cap on!"

This is so vague and
really meaningless that
no wonder small boys make fun of it!
But in mathematics we see that
THINKING consists of

(1) the judicious selection of
    an appropriate set of
    POSTULATES
    for the matter in hand,
    from which

(2) by means of LOGIC
    we derive valuable consequences.

Now of course
to make sure that
the postulate set is indeed
judiciously chosen —
"there's the rub" —
this requires genius!
And even a genius may
either make a mistake or
omit something (as Euclid did),
which may take
a couple of thousand years
to correct!
So you see it takes
REAL COURAGE
to be a mathematician —
to continue to construct systems,
but with the
UTMOST CAUTION,
and always re-examining them
and polishing them
to eliminate flaws,
for even what looks like

a TINY flaw
(like Euclid's, see p. 323)
may result in an unbearable
PARADOX (p. 322).

We must realize that
even to a dog
PARADOX is UNBEARABLE!
Thus Pavlov, in his experiments with
"conditioned reflexes",
found that
when a dog was conditioned to
react PLEASANTLY to a circle
and UNPLEASANTLY to an ellipse,
and then was shown
an ellipse that looked
so nearly like a circle that
he was terribly confused, —
the "paradox" was so overwhelming
that the poor dog
threw a violent fit
and forgot everything he had been
previously taught!
Is it any wonder then
that we humans too
CANNOT ENDURE PARADOX —
it tears the mind apart,
just as the body can be
torn apart by
pulling one arm in one direction
and the other arm in
the opposite direction —
real hard, I mean!

And now let us see,
in the next chapter,
what paradoxes arose in

the Theory of Transfinites,
and how they have been
eliminated.

But of course
you now know that
further paradoxes may arise
at any time,
not only in this theory,
but in any human theory —
this we MUST EXPECT,
but we must BRAVELY carry on
and continue
to GROW and IMPROVE what we have made,
for that is our
DESTINY —
or we shall perish.
That is why
the "lie", the "big lie",
the confusion it creates
is the most
DESTRUCTIVE force in existence,
much worse even than the atom bomb —
for the bomb produces
PHYSICAL destruction,
which is bad enough,
but
the big lie,
confusion,
contradiction,
paradox,
undermines our very ability to
use our minds,
our human-ness.
And consequently,
we really must not,
in the name of "free speech",

allow ourselves to be led by
anyone who uses the technique of
the "big lie" and its resultant confusion.
Yet this does NOT imply
the ELIMINATION of "free speech" —
far from it!
You can see this so clearly in
mathematics, where
CONTRADICTION (PARADOX)
is the greatest sin,
but this has NOT destroyed
the right to speak!
On the contrary,
mathematics is way out in front
as regards freedom of speech,
but it is LIMITED so as
NOT to permit PARADOX.
THERE
is FREEDOM to GROW, to PROGRESS,
THERE
is REAL THINKING,
THERE
lies the way to
SURVIVAL,
to LIFE.
In these dangerous times
we HAVE to take that as our
MODEL,
and not allow ourselves to be
MISLED by
a lot of "blah" about
the wrong, reckless, confusing kind of
"freedom",
which,
by the deadly weapon of the
"big lie",
will surely destroy us.

26 Paradoxes in the Theory of Transfinites and How They Have Been Conquered

Having seen what
a paradox means (p. 316),
and having seen some paradoxes
in mathematics,
as well as
HOW THEY WERE ELIMINATED,
you will not be surprised to know
that paradoxes arose also in
the Theory of Transfinites,
and that
they too have been eliminated —
though of course
others may, and probably will,
arise in the future (see p. 314).
And you now know also that,
in spite of this uncertainty about
the future,
the "show must go on"
if we, AS A RACE,
wish to LIVE and NOT DIE,
and that certainly
the story of mathematics is
THE SUCCESS STORY OF ALL TIME,
both in its THEORY
and in its
PRACTICAL APPLICATIONS.

Well, then,

let us get down to
"brass tacks",
and show you
at least ONE of the paradoxes in the
Theory of Transfinites.
One famous one was pointed out by
Bertrand Russell,*
another by Burali-Forti,*
another by Richards,*
et al.
Some of these are merely due
to a confusion of LANGUAGE
and therefore are
NOT really paradoxes inherent
in the system.
This is discussed thoroughly in
"Mathematical Logic" by
Hilbert and Ackermann,
which has recently (1950)
been translated into English
from the second German edition (1938),
and which is a most important book
on this subject.
But Russell's paradox is
of such a nature that
it was necessary to
re-examine and CHANGE some of
the basic ideas of the
Theory of Transfinites
in order to eliminate this paradox —
for eliminated it of course had to be,
as you now well know.

First let me state what

***See "Zehn Vorlesungen über die Grundlegung der Mengenlehre"**
by A. Fraenkel, pp. 21ff,
published in 1927.

Russell's paradox is,
and then merely indicate
what had to be done to eliminate it.
To go into full detail about
what had to be done
is beyond the scope of this little book,
for it would require a serious study of
both
the Theory of Transfinites
and
Modern Logic,
which could not be done in
a book of this kind,
intended for the intelligent layman
but not for professionals.

So let me proceed to give you
some idea of it
which, I hope, will be
intelligible and interesting to you:

First of all,
you already know that
the Theory of Transfinites
has to do with
SETS or AGGREGATES or CLASSES of
POINTS or NUMBERS or other "OBJECTS".
Now
Cantor thought,
and you will doubtless,
at this moment,
be inclined to agree with him,
that
it is conceivable to
think of a set of "objects"
no matter what property is used to
characterize these "objects".

But consider the following
interesting illustration of Russell's
of an IMPOSSIBLE set!
Imagine:

(1) A set which is
a member of itself, and

(2) A set which is
NOT a member of itself.

As an example of (1),
take "the set of all abstract concepts";
this is itself "an abstract concept"
and hence is a member of the set.*
As an example of (2),
take "the set of all human beings";
this SET is certainly
NOT a human being
and hence is NOT a member of this set.*
Finally

(3) consider the set consisting of
ALL sets of the kind described in
(2) above;
CALL THIS NEW SET M.

Now,
as Russell quite rightly argued,
the set M is
INCONCEIVABLE,
a SELF-CONTRADICTORY idea,
for:
obviously M cannot be of type (1) above,
i.e. M cannot be a member of M,
since M, by the definition in (3),

---

*See p. 21 of Fraenkel's book, referred to on p. 331.

contains ONLY sets of type (2);
on the other hand,
if M were of type (2),
it would belong in M by definition (3),
BUT
that says that
M belongs in M,
which it has no right to do
since, by definition (3),
M contains ONLY
sets that are NOT members of themselves,
so how can M contain itself?
Thus the set M
is NEITHER of type (1) NOR of type (2),
which is INCONCEIVABLE,
for any legitimate set
MUST be
either of type (1) or of type (2).
Thus Russell showed that
Cantor's idea of a set of
ANY "objects" whatever
was much too general,
for it is possible,
as you see,
to describe a set which is
an IMPOSSIBLE one!
And so
Russell and Whitehead
proposed the
"theory of types"
which would rule out
certain kinds of sets,
thus LIMITING the FREEDOM of
allowing
ANY kind of set of "objects",
as Cantor did.
But this "theory of types"

has turned out to be
unnecessarily complicated
and has been superseded by
what is known as the
"predicate calculus of order ω".
A detailed study of this calculus is
again beyond the scope of this book,*
but let me say something about it
which you will appreciate.

Now, you already know that
any mathematical system must
explicitly state
a SET OF POSTULATES for it
(a set of rules),
and then,
by means of LOGIC,
you derive, from this postulate set,
consequences or "theorems".
Now a good many people think that
LOGIC is just a little "common sense".
But you will be really surprised
to realize that

(1) the subject of LOGIC has
GROWN tremendously
since its simple beginnings in
naive "common sense";

(2) it has been put into
symbolic form
which employs symbols
(as mathematics does)
rather than "verbiage",

---

*But see the book by
Hilbert and Ackermann referred to
on p. 331.

and, as a result,
it has recently made such headway
(just as mathematics did
when it began to employ
symbols instead of words)
that

(3) there are now many DIFFERENT logics!*

Let me say just a word about
(1), (2), and (3) above,
and then indicate
what is the connection between
logic and
the elimination of the paradoxes from
the Theory of Transfinites.

In the first place,
may I say that
many, many people,
when they claim that
they are being "logical",
are merely using
the most naive "intuition"
(feminine or otherwise)
which is sometimes right
but very often wrong!
The first real logician,
the great Aristotle,
invented the idea of
the "syllogism",
by means of which

---

*Just as there are today many DIFFERENT geometries
(See my "Non-Euclidean Geometry"),
and many DIFFERENT algebras
(See "Fundamental Propositions
of Algebra" by E. V. Huntington
and my "Mits, Wits, and Logic").

one may derive correct conclusions
almost mechanically,
so that EVERYONE
(who took the trouble of
learning his method)
could really derive
correct logical results
instead of using
that old naive intuition
which can so often be so wrong!
You can see that
this "thought-machine" of Aristotle's
has been a most useful tool
for centuries.
But
the trouble with
Aristotelian Logic is that

(1) it applies only to
a limited number of situations,
situations in which
the answer is "Yes" or "No",
but of course you know perfectly well that
there are many situations
which are NOT of this kind,
and consequently
in which
Aristotelian logic
is useless;
and of course you also know
perfectly well that
there are many, many occasions
in which
Aristotelian logic is
FORCED upon people and
a "Yes" or "No" answer is demanded
though neither one would be correct!

(2) it has come down through the ages
in a very unwieldy form
with a truly terrible amount of
verbiage!
In recent times, however,
since it has been put into
symbolic form
(in about 1850,
by George Boole),
it has been boiled down
(believe it or not!),
to a mere THREE LINES,
with TWO LINES more added for
the subsequent contributions
to Aristotelian Logic made by
many later scholars,
making in all
a total of FIVE lines for
all of so-called
Traditional Logic —
but only when this is expressed in
MODERN symbolic form —
instead of the TOMES that have been
handed down to us by
generations of scholars.*
Furthermore,
by means of
MODERN SYMBOLISM,
many other logics
have now been developed
with which

*To get the details
see my little book:
"Mits, Wits, and Logic".
This "boiling down" is
really a marvellous process,
and shows the tremendous power of
SYMBOLIC or MATHEMATICAL
LOGIC.

many MORE situations can be
adequately treated.
Now, one of these new logics is the
"predicate calculus of order ω"*
(mentioned on p. 335)
which is appropriate for
the treatment of the
Theory of Transfinites.
This has now replaced the
theory of types of
Whitehead and Russell (see p. 334),
and, by means of this new logic,
the known paradoxes in
the Theory of Transfinites
HAVE BEEN SUCCESSFULLY ELIMINATED!
But of course we must never forget
that new paradoxes may arise
any day,
and then the mathematicians
will be obliged
to re-examine and improve the system
again,
as is usual when trouble arises —
rather than take a
defeatist attitude and
advocate throwing out
this beautiful and useful branch of
mathematics,
for this would indeed be like
"throwing out the lovely baby
with the dirty bath water" —
for SAM is no defeatist!

---

*For the serious student
I recommend again
the book by Hilbert and Ackermann
mentioned on p. 331.

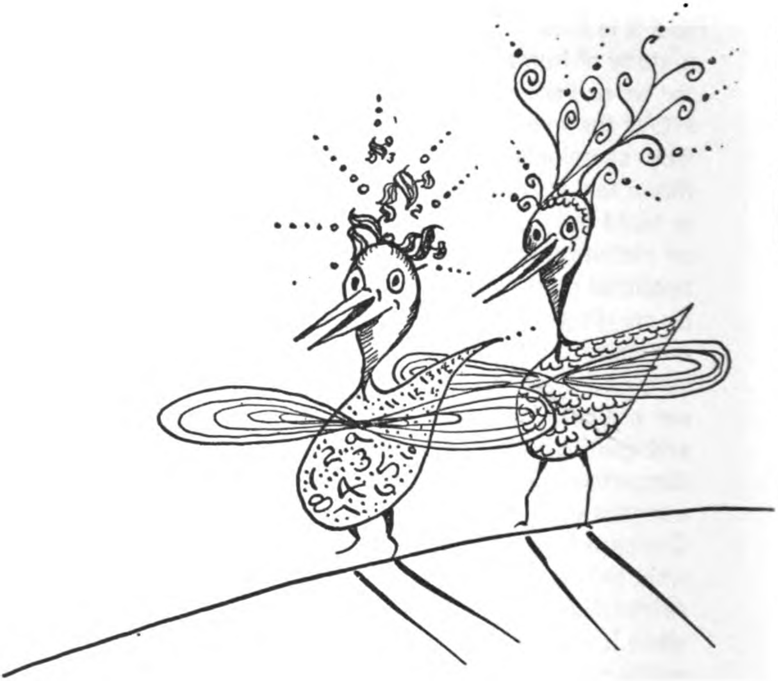
The Moral

Most people, as you know,
regard mathematics as
the outstanding "pain in the neck"
of their schooldays.
Or, at best, it is a
"necessary evil",
to be used,
under protest, in
figuring out their income tax,
in adding up long columns of numbers,
quickly and accurately,
and in
other such dreary tasks!

Now I wonder if
you agree with me,
after reading this little book,
that
nothing could be
further from the truth than
this false view of mathematics,
for it is really a domain full of
DARING,
FANTASY,
IMAGINATION,
FERTILITY,
CREATIVENESS,
unequalled even by
the finest in literature.

In mathematics we see
the human mind at work,

and it is thus
a study of human psychology,
for we can see
HOW the mind proceeds
from seemingly insignificant beginnings,
like a few postulates (see p. 138),
to build up
an elaborate and useful and
beautiful structure,
by creating a symbolism,
a language which is
so far superior to
our ordinary languages whose
ambiguities make them
dangerous and utterly inadequate as
a means of thought.
Compare for example
what happened to
Aristotelian Logic
when it was expressed in
mathematical symbolism (p. 338)
instead of in ordinary language,
how it developed into
a streamlined tool of thought
without which
we could not deal at all with
the paradoxes in the study of
INFINITY (see p. 339)
or with many other
TOUGH PROBLEMS.
And what would Einstein have done,
in his study of the Universe,
the General Theory of Relativity,
if he had not had
the marvellous
modern mathematical tool
known as the



Tensor Calculus* of
Levi-Civita!
Perhaps he would have been forced to
create this branch of mathematics
himself,
just as Newton did in the case of
his well-known
Differential and Integral Calculus
which he then used in
the study of the physical world.
But POWERFUL tools they must have,
whoever creates them!
Even the greatest mind,
like that of Einstein,
which is second to none
all through the ages,
still needs
ADEQUATE TOOLS for thought.
And this is just what
ordinary language lacks —
for there we are trying to fly
by flapping our arms
but never even get to
take off!

And are we not in desperate need of
a tool for
thinking about
human relations?
And does not
mathematics shed light even here?
Can we not,
by a study of mathematics,
get clearer ideas about

---

*See a brief account of it in
my little book:
"The Einstein Theory of Relativity."

"freedom" for instance?
Is it not imperative that we specify
freedom for WHAT?
Is freedom unlimited
allowed in mathematics?
Note the paradox which arose in
Cantor's Theory (p. 334)
when TOO MUCH FREEDOM
was permitted in the
concept of a "set"!
How does mathematics attain
a maximum of "healthy" freedom (p. 15)
without permitting
destructive paradoxes?

In short,
can we not learn,
from mathematics,
how to use our SAM
effectively and constructively
in dealing with
ourselves and our environment?

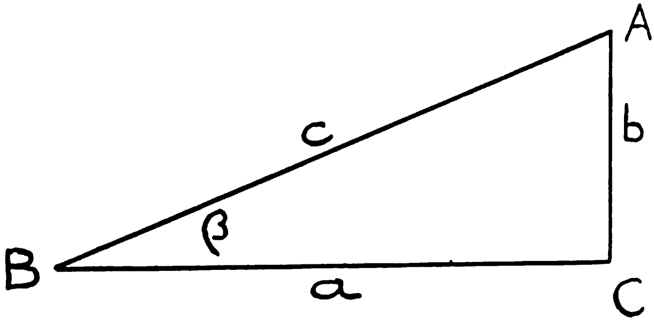
Personally I find
mathematics most helpful in
tackling daily problems;
I am convinced that the
honesty and patience,
for example,
without which
mathematics could not develop,
are also essential as elements in
a philosophy of living.
And I cannot go along with
those who say that
mathematics and science
do not furnish

human "values",
that they are
AMORAL,
not caring whether their products,
like atomic energy,
are used for
constructive or destructive purposes!
This seems to me to be
quite false,
for atomic energy itself
could not have been discovered
without
honesty and patience —
I am not implying that
this is ALL that was needed,
but I am merely saying that
whatever else was needed,
great creative ability for instance,
could not have functioned in
conjunction with
the BIG LIE,
the PARADOX!
That is to say,
in mathematics
CRIME DOES NOT PAY.

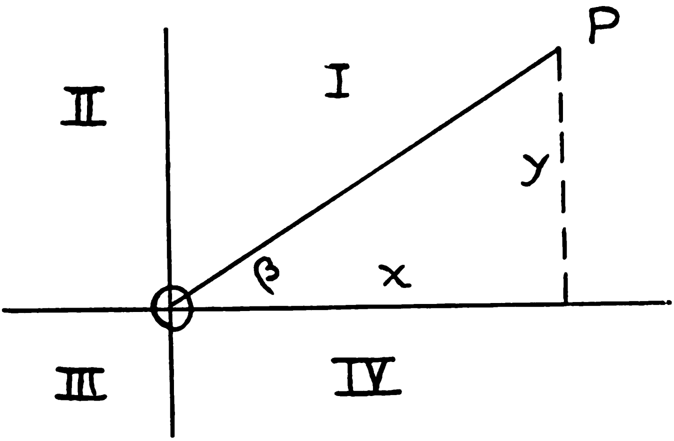
And therefore it seems to me
that the morality
inherent in the development of
mathematics (and science) itself
is a human "value"
which we should then APPLY
IF WE WISH TO BE
AS SUCCESSFUL
in our philosophy of living
as in the development of
mathematics.

Appendix A: Definitions of the Word "Tangent"

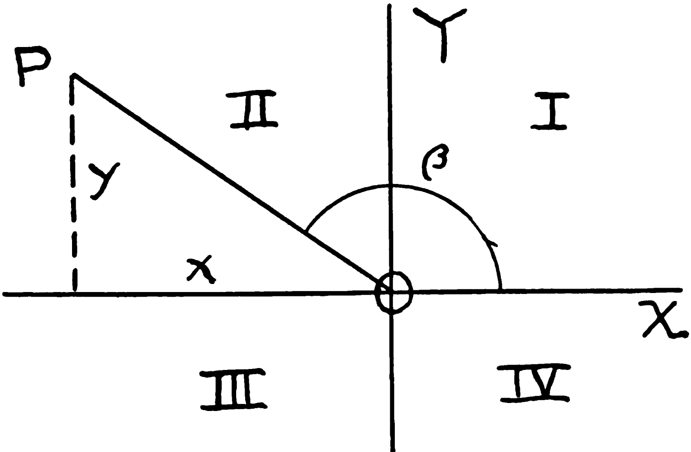
(1) The "tangent" of an angle, β,



is defined in the right triangle ABC
(right angle at C)
as the ratio b/a,
that is, the ratio of
the side, b, opposite angle β to
the side, a, adjacent to the angle;
or (see the graphs on p. 347)
it is MORE GENERALLY defined as
the ratio of the
"ordinate", y,
of point P
to the "abscissa", x, of P, thus:
$\tan \beta = y/x$.
I say "more generally" because
this second way of defining it

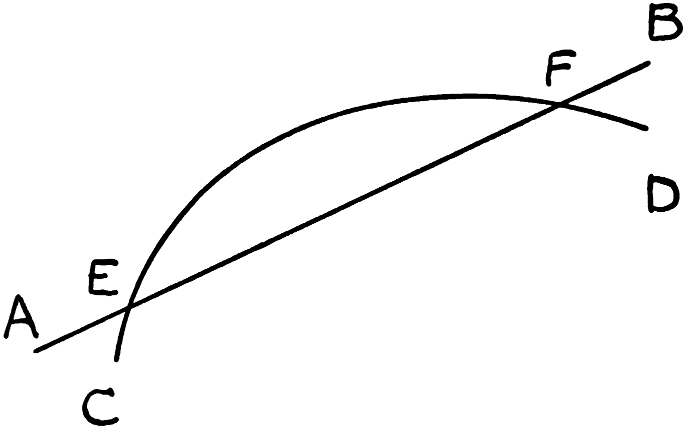

P
I
II
y
β
x
III
IV

can apply NOT ONLY when
β happens to be in quadrant I,
i.e. when *β* is an acute angle,
but also when
it is in ANY one of the four quadrants:

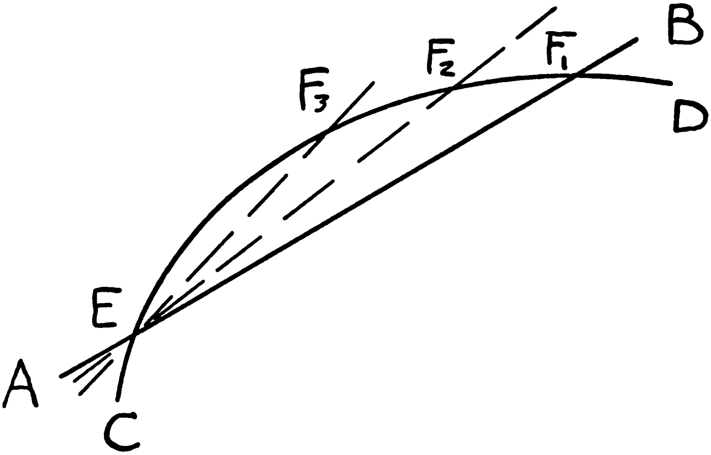

Y
P
II
I
y
β
x
X
III
IV

In the second diagram on p. 347
β is an obtuse angle,
i.e. it is between 90° and 180°
and therefore lies in the SECOND quadrant
and yet again
$\tan \beta = y/x$.
Of course in this case,
x, being to the LEFT of the origin, O,
is NEGATIVE,
and y, being ABOVE the X-axis,
is positive,
and therefore their ratio, y/x,
is NEGATIVE,
so that the tangent of any angle in
the second quadrant
is always negative.
Similarly for quadrants III and IV:
but see if you can figure out yourself
the sign of the tangent of
an angle in each of these quadrants
(remember that the sign of the ratio of
two quantities having LIKE signs —
both positive or both negative —
is POSITIVE,
and that the ratio of
two quantities having UNLIKE signs —
one positive and the other negative —
is NEGATIVE).

(2) Another use of the word "tangent":
if a straight line, AB,
crosses a curve, CD,
in two points, E and F,
as in the first diagram on p. 349,
then AB is called a "secant";
now if AB is revolved around E,
so that F comes

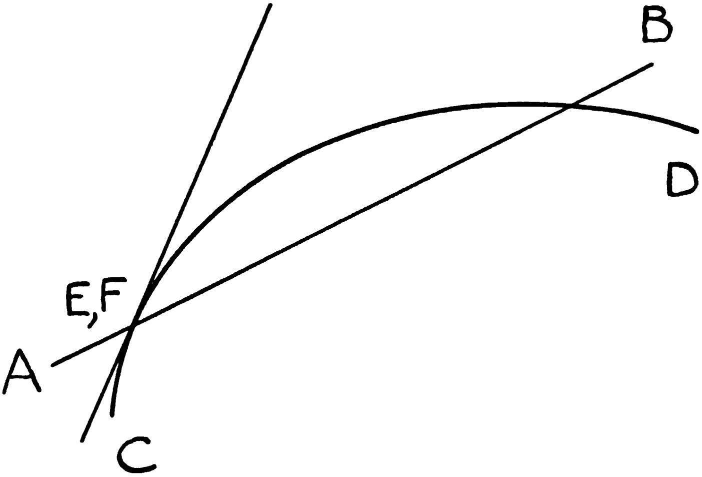


nearer and nearer to E,
as is shown by the dotted lines:


B
F_2
F_1
F_3
D
E
A
C

then,
when F finally reaches E,
the "secant" becomes a "tangent"

thus:



And so you see from (1) and (2) that the word "tangent" has two different meanings:

(1) the "tangent" of an angle
is one thing
and

(2) a "tangent" line is quite another.

And I trust you will now not be confused by the statement: the "slope" of a curve at a given point, P, is the "tangent" of the ANGLE β, formed by the LINE which is "tangent" to the curve at P and a line which is parallel to OX (see p. 240).

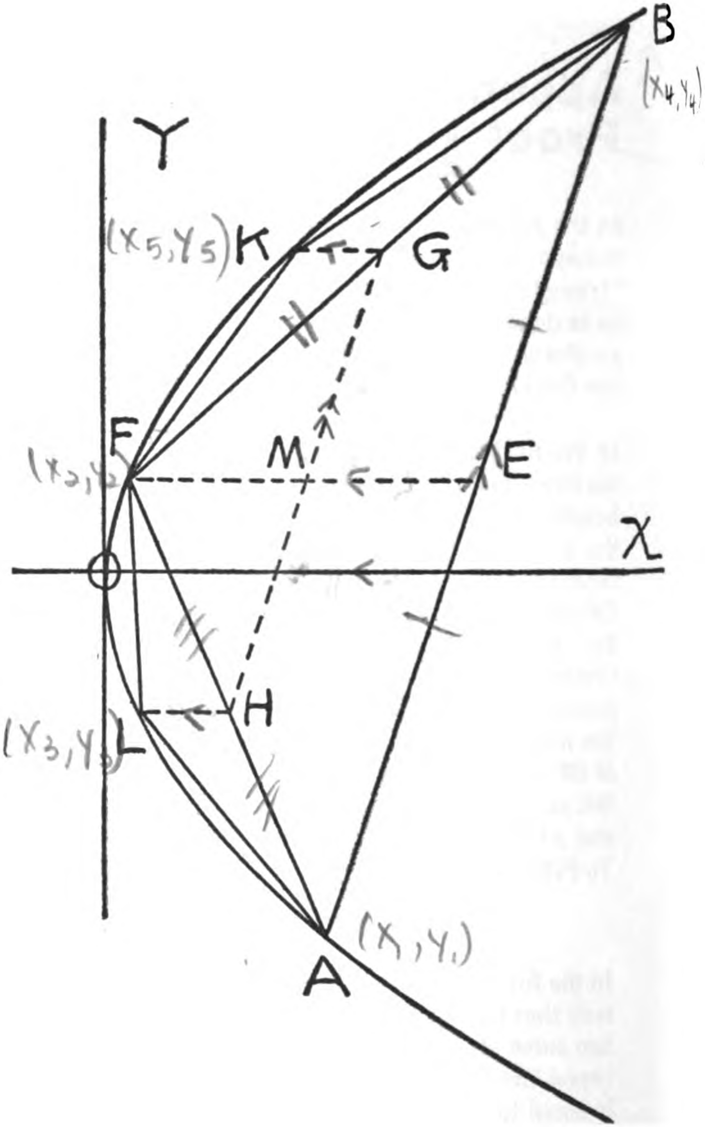
Appendix B: Proof Promised on p. 260

In the following I shall use
the symbol △ to represent the word
"triangle",
as is done in text-books on geometry —
so please do not mistake it here for
the Greek letter "delta" (p. 229).

In the next diagram (p. 352)
we have given the parabolic segment
bounded by the chord AB and
the parabolic arc AOB.
Given also E the mid-point of AB,
EF parallel to OX,
thus forming △ABF;
similarly,
points G and H are
the mid-points
of BF and AF, respectively.
GK and HL are both parallel to OX,
and △s FKB and FLA are thus formed.
To PROVE now that

$$\triangle ABF = 4(\triangle FKB + \triangle FLA).$$

In the first place,
note that GH joins the mid-points of
two sides of △ABF,
hence GH is parallel to AB and
is equal to (1/2) AB in length.
Therefore △FGH is similar to △FBA


B
(x_4, y_4)
Y
(x_5, y_5) K
G
F
(x_2, y_2)
M
E
X
O
H
(x_3, y_3) L
(x_1, y_1)
A

(since their corresponding angles
are equal,
GH being parallel to BA).
Hence $\triangle FBA = 4\triangle FGH$
(since they are similar triangles and
their corresponding sides are
in the ratio 2:1).*
And therefore we must prove that

$$\triangle FGH = \triangle FKB + \triangle FLA.$$

Now since KG, FM, and LH are
medians in ⟁FKB, FGH, and FLA,
respectively
(a "median" is a line from
any vertex of a triangle to
the mid-point of the opposite side),
hence

$$\begin{aligned}\triangle FKG &= (1/2)\triangle FKB\\ \triangle FGM &= (1/2)\triangle FGH\\ \triangle FLH &= (1/2)\triangle FLA\end{aligned}$$

because a median divides a △ into
two ⟁ having equal areas.
And so we must prove that

$$\triangle FGM = \triangle FKG + \triangle FLH.$$

Now $\triangle FKG/\triangle FGM = KG/FM \qquad (1)$

(since 2 ⟁ having the same altitude
are to each other as their bases)*
and

$$\triangle FLH/\triangle FHM\ (= \triangle FGM) = LH/FM. \qquad (2)$$

---

*Look up these theorems in any book on
elementary plane Euclidean Geometry.

Adding (1) and (2) we get

$$(\triangle FKG + \triangle FLH)/\triangle FGM = (KG + LH)/FM.$$

Hence we must prove that

$$FM = KG + LH.$$

Let the coordinates of A be (x_1, y_1)
“ “ “ “ F “ (x_2, y_2)
“ “ “ “ L “ (x_3, y_3).

Then $LH = [(x_1 + x_2)/2] - x_3$.

Let the coordinates of B be (x_4, y_4)
“ “ “ “ K “ (x_5, y_5).

Then $KG = [(x_2 + x_4)/2] - x_5$.

And $FM = 1/2FE = (1/2)[\{(x_1 + x_4)/2\} - x_2]$.

Thus we must prove that

$(1/2)[\{(x_1 + x_4)/2\} - x_2] =$
$[(x_2 + x_4)/2] - x_5 + [(x_1 + x_2)/2] - x_3$.

Now, since the parabola is $y^2 = 4x$,
therefore

$x_5 = y_5^2/4 = (1/4)[(y_2 + y_4)/2]^2$
$= (1/16)\ (y_2^2 + 2y_2y_4 + y_4^2)$

$\therefore x_5 = (1/16)(4x_2 + 2y_2y_4 + 4x_4)$.

And $x_3 = y_3^2/4 = (1/4)[(y_1 + y_2)/2]^2$
$= (1/16)(y_1^2 + 2y_1y_2 + y_2^2)$

$\therefore x_3 = (1/16)(4x_1 + 2y_1y_2 + 4x_2)$.

Hence we must prove that

$(x_1 + x_4)/4 - x_2/2 =$
$(x_2 + x_4)/2 - x_2/4 - (y_2y_4)/8 - x_4/4 +$
$(x_1 + x_2)/2 - x_1/4 - (y_1y_2)/8 - x_2/4$

or

$$-(2x_2 + 2x_2 - x_2 + 2x_2 - x_2)/4$$
$$= (-y_2/8)(y_1 + y_4)$$

or

$$x_2 = (y_2/8)(y_1 + y_4)$$

or

$$y_2^2/4 = y_2(y_1 + y_4)/8$$

or

$$2y_2 = y_1 + y_4$$

or

$$y_2 = (y_1 + y_4)/2.$$

Q.E.D.

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Appendix C1: An Infinite Set Is Equivalent to a PART of Itself*

In the figure on p. 110
let $AB = z$, where z has the range

$$0 \leqq z \leqq 1,$$

i.e. z may have any real value between
0 and 1, inclusive;
and let $A'B' = z'$,
where z' has the range
$0 \leqq z' \leqq (1/2)$;
and let
$zz' + z' - z = 0$
from which we get

$$z' = z/(1 + z)$$

and $$z = z'/(1 - z').$$

From these equations we can see that
when $z = 0$, then $z' = 0$
and when $z = 1$, then $z' = (1/2)$;
and
when $z' = 0$, then $z = 0$
and when $z' = (1/2)$, then $z = 1$;
and, finally,
for every real value of z between 0 and 1,
there is a corresponding real value of
z' between 0 and 1/2,
and conversely,
for every real value of z'
between 0 and 1/2,

---

*See "Actual Infinity" by
S. A. Bogomolov (1934), p. 10.

there is a corresponding real value of
z between 0 and 1.
Thus there is a 1-1 correspondence
between the real numbers
in the two intervals
from 0 to 1 and
from 0 to 1/2.

And, of course,
the same could be shown for
the real numbers in
ANY two intervals.

Appendix C2: The Set of Real Numbers Between 0 and 1 Is Equivalent to the Set of ALL Real Numbers

Let $$z' + z'e^z - e^z = 0, \qquad (1)$$

from which, solving for z',
we get $z'(1 + e^z) = e^z$

or $$z' = e^z/(1 + e^z)\,; \qquad (2)$$

and, solving (1) for e^z,
we get

$e^z - z'e^z = z'$

or $e^z(1 - z') = z'$

or $e^z = z'/(1 - z')$;

hence $$z = \log z'/(1 - z'). \qquad (3)$$

Dividing the numerator and denominator
of the fraction in (2) by e^z,
(2) becomes
$$z' = 1/[(1/e^z) + 1]. \qquad (4)$$

If, now, z has the range $-\infty < z < +\infty$,
then, as z approaches $-\infty$,
$1/e^{-\infty} = e^{\infty}$ approaches $+\infty$,*
hence, in (4), z' approaches 0;
and, as z approaches $+\infty$,
z' approaches 1,

---

*Remember, from elementary algebra,
that
$$a^{-n} = 1/a^n.$$

so that the range of z' is
$0 < z' < 1$.
And for each value of z from $-\infty$ to $+\infty$
there will be a definite
corresponding value of z' from 0 to 1;
and, conversely,
if z' represents all the
real numbers in (a) on p. 116,
that is,
if z' has the range
$0 < z' < 1$,
then, from equation (3) above
we easily see that
as z' approaches 0
z will approach $-\infty$;
and as z' approaches 1,
then z approaches $+\infty$;
and
for each value of z' from 0 to 1,
there is a corresponding value of z
from $-\infty$ to $+\infty$.
Thus the sets (a) and (b) on p. 116
are in 1-1 correspondence
and are therefore
"equivalent" and hence
the "power" of
ALL real numbers is
also c*.

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*See p. 341 of
"A Survey of Modern Algebra" by
Birkhoff and MacLane.

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